Math 381: First Exam

September 26, 2003

Solutions

Problem 1: The equations are second order, fourteenth order and first order, respectively.

Problem 2: We compute the derivatives

$$\frac{\partial^2 y}{\partial x^2} = f''(x+at) - g''(x-at),$$
$$\frac{\partial^2 y}{\partial t^2} = a^2 f''(x+at) - a^2 g''(x-at).$$

So then it is not hard to check that

$$\frac{\partial^2 y}{\partial t^2} - a^2 \frac{\partial^2 y}{\partial x^2} = a^2 f''(x+at) - a^2 g''(x-at) - a^2\left(f''(x+at) - g''(x-at)\right) = 0.$$

Problem 3: We need something which is linear in the first paritals, but not jointly linear in the first partials and z. So,

$$z\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0.$$

Problem 4: This equation is quasilinear, so we use the method of characteristics. The CVF is $V = (1, -y, e^{-z})$, and the characteristic ODE's are

$$\frac{dx}{1} = \frac{dy}{-y} = \frac{dz}{e^{-z}}.$$

We solve $dx = e^z\ dz$ with $x + C_1 = e^z$, and $-dx = \frac{dy}{y}$ with $-x + C_2 = \ln(y)$. So, writing C_1 as an arbitrary function of C_2, we obtain

$$z = \ln\left(x + f(x + \ln(y))\right).$$

Problem 5:

a) The characteristic vector field is $V = (y, -x, 0)$, so the characteristic ODE's are

$$\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{0}$$

The last bit means that $dz = 0$, so $z = C_1$ is a constant. We then solve the other equation

$$x\ dx = -y\ dy$$

to get $x^2 + y^2 = C_2$. Thus, the characteristics are defined by

$$\left\{ \begin{array}{c} z = C_1 \\ x^2 + y^2 = C_2 \end{array} \right\}, \qquad \text{where } C_1 \text{ and } C_2 \text{ are constants.}$$

These curves are circles lying in the horizontal planes parallel to the xy-plane and with centers on the z-axis.

b) We check the condition for freeness of the curve σ by computing

$$P(\sigma_1(t), \sigma_2(t), f(t)) \cdot \sigma_2'(t) - Q(\sigma_1(t), \sigma_2(t), f(t)) \cdot \sigma_1'(t) = (\sigma_2(t)) \cdot (0) - (-\sigma_1(t)) \cdot (1) = t.$$

We are only interested in the positive x-axis, where $t > 0$, so this quantity does not vanish, and the curve is free.

c) Using the work in (a), we see that the general solution can be written as $z = f(x^2 + y^2)$. To meet our initial conditions, we must have $x^2 = f(x^2 + 0)$, so $f(u) = u$ is the identity function. This means that our integral surface is defined by the equation $z = x^2 + y^2$. This surface is a paraboloid of revolution, where the axis of revolution is the z-axis.

d) There are infinitely many solutions. Namely, we can use $z = f(x^2 + y^2)$ where f is an arbitrary function which satisfies $f(5) = 25$.

Problem 6: This was a homework problem. You can find the answer in the solutions to homework #2.

Problem 7: The correct statement is

Let f be a 2π periodic function for which both f and f^2 are integrable. Then

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x) \sin\left((n + \frac{1}{2})x\right) = 0.$$

Problem 8: I'm not going to write an example essay, but a good answer should include a discussion of convergence at a point of continuity and at a point of discontinuity. Also, it should have something about uniform convergence for broken line functions, and something about the Gibbs phenomenon. It is important to make precise statements of the results we have from class.

It might also be nice to include something about Césaro summability and Féjer's theorem.