Math 381: Second Exam

Due: Monday, November 10, 2003 at the beginning of class.

This exam is self-timed. You have a maximum of 3 hours. You may consult your textbook or your class notes, but you may not discuss the content of this exam with anyone but your instructor.

Extra instructions: Answer each question completely and carefully. Please start each new problem on a new page and number your pages. Show all of your work. You may quote something we did in class or on homework, but do so precisely. Please return this signed cover sheet with your exam paper.

Name:

Student ID:

On my honor, I have neither given nor received any aid on this examination.

Signed:____

Problem 1: Find a function u which is harmonic in the disk $\mathcal{D} = \{(x, y) \mid x^2 + y^2 \leq 1\}$ and equal to $g(x, y) = x^2$ on the boundary circle $\mathcal{C} = \{(x, y) \mid x^2 + y^2 = 1\}$.

Problem 2: In class we considered the problem of describing the motion of a plucked string which has length π. If we add a term to represent a damping force which is proportional to the velocity of the string (like air resistance, or some other complicating factor), we get a new equation for the "vertical displacement" $y(x, t)$ of the string from horizontal (rest position) at a point $x \in (0, \pi)$ and time $t \geq 0$:

$$\frac{\partial^2 y}{\partial t^2} + k\frac{\partial y}{\partial t} = a^2 \frac{\partial^2 y}{\partial x^2}.$$

Here, the constants a and k depend on the physical properties of the string and the damping force.

Perform the separation of variable technique on this problem to obtain the ordinary differential equations that one must solve. Do not solve these new equations.

Problem 3: Suppose that the temperature T on the edge $\mathcal{C}$ of the unit disk $\mathcal{D}$ is held steady so that $T(x, y) = yx^3$ degrees Fahrenheit for $(x, y) \in \mathcal{C}$. What is the temperature at the center of the disk?

Problem 4: Given a continuous function $f(x)$, let us agree to denote the new function

$$x \mapsto \int_0^x f(t)dt$$

by $f^{(-1)}(x)$. Then we can define $f^{(-n)}(x)$ by applying this operation n times: $f^{(-n)}$ is obtained from f by integrating n times, and choosing the constants of integration at each step so that $f^{(-n)}(0) = 0$ for each n. Recall that the Fundamental Theorem of Calculus says that $f^{(-n)}$ has n continuous derivatives, that is, $f^{(-n)}$ is a C^n function.

Now consider the function $f(x) = (x^2 - 1)^{-6}$. What is the coefficient of $P_6(x)$ in the Legendre series for $f^{(-6)}$?

Problem 5: Let a_k be the kth Legendre series coefficient of a function $f(x)$ which is continuous on $[-1, 1]$. Evaluate the sum

$$\sum_{k=0}^{\infty} \frac{2}{2k+1} a_k^2$$

in terms of the function f. (*Hint: What is the corresponding thing for Fourier series?*)

Problem 6: Let $P_n(x)$ denote the nth Legendre polynomial. What is the Legendre series of the function $f(x) = xP_{101}(x)$?