Math 381: Second Exam

November 5-10, 2003

Solutions

Problem 1: We rewrite our problem in polar coordinates because these are adapted to the disk $\mathcal{D}$. Then our boundary condition is $g(1,\theta) = (\cos\theta)^2$. Following the developments in class, we know that the solution to Laplace's equation in $\mathcal{D}$ which meets this boundary condition is

$$u(r,\theta) = \frac{a_0}{2} + \sum_{i=1}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) r^n$$

where the constants $a_0/2$, a_n and b_n are chosen to be the Fourier series coefficients of $f(\theta) = g(1,\theta) = (\cos\theta)^2$. Using a little bit of trigonometry, we see that the Fourier series of f is

$$f(\theta) = (\cos\theta)^2 = \frac{1}{2} + \frac{1}{2}\cos(2\theta).$$

(This is just the half-angle identity. You can get this lots of ways, even by integrating for Fourier coefficients. I remembered it as the $k=1$ special case of the identity $\cos(k+1)\theta + \cos(k-1)\theta = 2\cos\theta\cos k\theta$ which we used in class while working out the Poisson integral formula.) So the answer is

$$u(r,\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta) r^2.$$

Unwinding the half-angle identity, we see that this is the same as

$$u(r,\theta) = \frac{1}{2} + \frac{r^2}{2}\left(2(\cos\theta)^2 - 1\right) = \frac{1-r^2}{2} + (r\cos\theta)^2.$$

In rectangular coordinates, this is

$$u(x,y) = \frac{1 + x^2 - y^2}{2}.$$

Problem 2: We assume that our solution has the form of a product $y(t,x) = T(t)\cdot F(x)$. Then the differential equation requires that the functions T and F satisfy

$$T''\cdot F + kT'\cdot F = a^2 T\cdot F''.$$

We rearrange this to read

$$\frac{T'' + T'}{T} = a^2\frac{F''}{F}.$$

Since the right hand side is independent of t and the left hand side is independent of x, we conclude that this above quantity should be constant. If we write C for this constant, we need to solve the following two ordinary differential equations:

$$T'' + k\cdot T' - C\cdot T = 0, \quad \text{and} \quad a^2F'' - C\cdot F = 0.$$

Problem 3: Translating to polar coordinates (r, θ), we see that our boundary condition is $T(r, \theta) = r \sin\theta (r\cos\theta)^3$ on the boundary. But the boundary is the curve $r = 1$. So our boundary condition is $f(\theta) = T(1, \theta) = \sin\theta(\cos\theta)^3$. Using the Poisson integral formula for harmonic functions in the disk, we get that

$$T(0, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \sin\phi(\cos\phi)^3 \cdot \frac{1-0}{1-0+0}\, d\phi = 0.$$

So the answer is that the temperature at the center of the disk is zero degrees Fahrenheit.

Problem 4: Using Rodrigues' formula and the theorem following it in class (Theorem 5), we have that

$$a_6 = \frac{13}{2} \cdot \frac{(-1)^6}{2^6 \cdot 6!} \int_{-1}^{1} \frac{d^6}{dx^6}(f^{(-6)})(x) \cdot (x^2-1)^6 \, dx.$$

But by the construction, the sixth derivative of f is equal to $x \mapsto (x^2-1)^{-6}$. Therefore, we get that

$$a_6 = \frac{13}{2} \frac{1}{2^6 \cdot 6!} \int_{-1}^{1} (x^2-1)^{-6}(x^2-1)^6 \, dx = \frac{13}{2^6 \cdot 6!}$$

Problem 5: Since f is continuous on the closed interval $[-1, 1]$, it is certainly bounded and has both f and f^2 integrable. By Parseval's theorem, we see that the answer is

$$\sum_{k=0}^{\infty} \frac{2}{2k+1} a_k^2 = \int_{-1}^{1} f(t)^2 \, dt.$$

Problem 6: Using the recurrence formula for Legendre polynomials, we see that the answer is

$$xP_{101}(x) = \frac{102}{203} P_{102}(x) + \frac{101}{203} P_{100}(x).$$