Solutions to Homework # 1
Math 381, Rice University, Fall 2003

***Hildebrand*, Ch. 8, # 1**:
Part (a). We compute

$$\frac{\partial z}{\partial x} = f(x+y) + (x-y)f'(x+y)$$
$$\frac{\partial z}{\partial y} = -f(x+y) + (x-y)f'(x+y).$$

Subtracting, we eliminate f'...

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 2f(x+y).$$

Substitute in from the original expression to get

$$(x-y)\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) - 2z = 0.$$

Part (b). We make a convenient change of variables:

$$u = ax + by, \qquad v = cx + dy.$$

In these variables, our expression is $z = f(u) + g(y)$ which has accompanying PDE

$$\frac{\partial^2 z}{\partial u \partial v} = 0.$$

We need to translate this back to x, y-coordinates! Note that if $\varepsilon = (ad - bc)$, we have

$$x = \varepsilon^{-1}(du - bv), \qquad y = \varepsilon^{-1}(-cu + av).$$

(This should look familiar if you have studied 2x2 matrices and their inverses.) This allows us to compute that (hang on, it's not that bad)

$$\begin{aligned}
0 = \frac{\partial^2 z}{\partial u \partial v} &= \frac{\partial}{\partial u}(\frac{\partial z}{\partial v}) = \frac{\partial}{\partial u}(\frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}) \\
&= \frac{\partial}{\partial u}(\frac{\partial z}{\partial x}(-\varepsilon^{-1}b) + \frac{\partial z}{\partial y}(\varepsilon^{-1}a)) \\
&= -b\varepsilon^{-1}\left(\frac{\partial}{\partial x}(\frac{\partial z}{\partial x})\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}(\frac{\partial z}{\partial x})\frac{\partial y}{\partial u}\right) + a\varepsilon^{-1}\left(\frac{\partial}{\partial x}(\frac{\partial z}{\partial y})\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}(\frac{\partial z}{\partial y})\frac{\partial y}{\partial u}\right) \\
&= -bd\varepsilon^{-2}\frac{\partial^2 z}{\partial x^2} + (ad+bc)\varepsilon^{-2}\frac{\partial^2 z}{\partial x \partial y} - ac\varepsilon^{-2}\frac{\partial^2 z}{\partial y^2}
\end{aligned}$$

Now clearing some common factors we get

$$bd\frac{\partial^2 z}{\partial x^2} - (ad+bc)\frac{\partial^2 z}{\partial x \partial y} + ac\frac{\partial^2 z}{\partial y^2} = 0.$$

Part (c). Here we use a change of variables:

$$u = ax + by, \qquad v = x$$

and proceed as above. Our expression looks like

$$z = f(u) + vg(u)$$

So, we differentiate to find

$$\begin{aligned}\frac{\partial z}{\partial u} &= f'(u) + vg'(u)\\ \frac{\partial z}{\partial v} &= g(u)\\ \frac{\partial^2 z}{\partial v^2} &= 0\end{aligned}$$

This last equation is our PDE. But we must write it in the x, y-coordinates. So by a computation analogous to the part (b), we find that our desired PDE is

$$b^2\frac{\partial^2 z}{\partial x^2} - 2ab\frac{\partial^2 z}{\partial x \partial y} + a^2\frac{\partial^2 z}{\partial y^2} = 0.$$

Which is what we wanted.

From now on, I will use the notation z_x for $\frac{\partial z}{\partial x}$. It saves typing, and matches Hildebrand's notation.

Hildebrand, **Chapter 8, #2**:

Part (a). Differentiating the relation $z = f(\psi)$ with respect to x and y, we find

$$z_x = f'(\psi)\psi_x, \qquad z_y = f'(\psi)\psi_y$$

Which means that

$$f'(\psi) = \frac{1}{\psi_x}z_x = \frac{1}{\psi_y}z_y,$$

which implies that

$$\psi_y z_x - \psi_x z_y = 0. \tag{1}$$

Part (b). So, again we use the chain rule to compute the derivatives of z with respect to x and y while considering s and t as intermediate functions. We get

$$\begin{aligned}z_x &= z_s s_x + z_t t_x \quad &= z_s s_x + z_t\psi_x,\\ z_y &= z_s s_y + z_t t_y \quad &= z_s s_y + z_t\psi_y.\end{aligned}$$

If we substitute these into 1, we see

$$\begin{aligned}0 &= \psi_y\left(z_s s_x + z_t\psi_x\right) - \psi_x\left(z_s s_y + z_t\psi_y\right)\\ &= z_s\left(\psi_y s_x - \psi_x s_y\right).\end{aligned} \tag{2}$$

Which is what we wanted. To see that the most general solution of the equation given in the problem has the form $s = f(\psi)$, note that the two functions $t = \psi(x, y)$ and $s = s(x, y)$ are independent exactly when $(\psi_y s_x - \psi_x s_y) \neq 0$. Therefore, equation 2 is equivalent to $\dfrac{\partial z}{\partial s} = 0$. The *general* solution of this differential equation is clearly $z = f(t) = f(\psi)$ where f is an arbitrary function.

Hildebrand, **Chapter 8, # 3**.
If $z = \psi(x, y)$ is a solution of the equation given in the problem, then we see that

$$P \cdot \psi_x + Q \cdot \psi_y = 0.$$

Which translates to

$$Q = -P \cdot \frac{\psi_x}{\psi_y}.$$

If we substitute this back into the given PDE, we get the equivalent equation

$$P \frac{\partial z}{\partial x} - P \frac{\psi_x}{\psi_y} \frac{\partial z}{\partial y} = 0,$$

simplifying, we find

$$P \left[\psi_y \frac{\partial z}{\partial x} - \psi_x \frac{\partial z}{\partial y} \right] = 0.$$

So as long as the function P is not identically zero, we see that our equation is equivalent to the one considered in problem 2. By the result of problem 2, the most general solution has the form $z = f(\psi)$ where f is an arbitrary function.

Hildebrand, **Chapter 8, # 4**:
We see by differentiating that

$$z_x = \phi f'(\psi) \psi_x + \phi_x f(\psi),$$
$$z_y = \phi f'(\psi) \psi_y + \phi_y f(\psi).$$

We eliminate $f'(\psi)$ to get the equation

$$\psi_y \frac{\partial z}{\partial x} - \psi_x \frac{\partial z}{\partial y} = (\psi_y \phi_x - \psi_x \phi_y) f(\psi)$$

Since we can then substitute $f(\psi) = z/\phi$, we are done.

Hildebrand, **Chapter 8, #5**:
Part (a). The characteristic vector field is $V = (a, b, c)$. So we get characteristic ODE

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c}.$$

Of these, we pick $a \cdot dy - b \cdot dx = 0$ and $a \cdot dz - c \cdot dx = 0$. These have solutions $ay - bx = C_1$ and $az - cx = C_2$, respectively. Therefore, we can write our most general solution in the form

$$az - cx = C_2 = f(C_1) = f(ay - bx)$$

or,

$$z = \frac{c}{a}x + f(ay - bx)$$

where f is an arbitrary function of class C^1.
Part (b). The CVF is $V = (a, b, cz)$. The associated ODE is

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{cz}.$$

Again, we solve a pair of these to get $ay - bx = C_1$ and $cx - a \ln z = C_2$. So our most general solution is

$$cx - a \ln z = f(ay - bx)$$

or, equivalently,

$$z = e^{(cx/a - f(ay - bx))},$$

where f is an arbitrary C^1 function.
Part (c). The CVF is $V = (y, -x, 0)$. The associated ODE is

$$\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{0}$$

So we solve these with $z = C_2$ and $x^2 + y^2 = C_1$. The most general solution is then

$$z = f(x^2 + y^2)$$

where f is an arbitrary C^1 function.
Part (d). The CVF is $V = (1, 1, -2xz)$. The associated ODE is

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{-2xz}.$$

We use the equations $dx = dy$ and $dx = \dfrac{dz}{-2xz}$. These have solutions $y - x = C_1$ and $x^2 + \ln z = C_2$. So, the general solution to our PDE is

$$z = e^{-x^2 + f(y-x)},$$

where f is an arbitrary C^1 function.
Part (e). The CVF is $V = (x, -y, z)$. The associated ODE is

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{z}$$

We solve the equation $\frac{dx}{x} = \frac{dy}{-y}$ with $\ln(xy) = C_1$ and the equation $\frac{dx}{x} = \frac{dz}{z}$ with $\ln(z/x) = C_2$. We then get that the general solution is (after clearing the logarithms)

$$z = xf(xy),$$

where f is an arbitrary C^1 function.
Part (f). The CVF is $V = (x^2, y^2, z^2)$. The associated ODE is

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$$

We can solve a pair of these with the relations $C_1 = \frac{1}{x} - \frac{1}{y}$ and $C_2 = \frac{1}{z} - \frac{1}{x}$. Which means that the general solution to our PDE has the form

$$z = \frac{1}{\frac{1}{x} + f(\frac{1}{x} - \frac{1}{y})},$$

where f is an arbitrary C^1 function.
Hildebrand, **Chapter 8, #8**:
Part (a). The characteristic vector field is the constant field $V = (1, 1, 1)$. Therefore, all the characteristic curves are lines with this direction vector. The final claim follows by plugging in the given point.
Part (b). The curve through $(0, y_0, z_0)$ is given by $x = y - y_0$, $x = z - z_0$. If the initial point is to be chosen on the curve $z = y^2$, $x = 0$, then we get equations $y - x = y_0$, $z - x = z_0 = y_0^2$, as desired.

To get the equations of the surface which is traced out, note that $z = x + y_0^2 = x + (y - x)^2$.
Part (c). The given surface "obviously" contains the curve. As for being an integral surface, we note that

$$\frac{\partial z}{\partial x} = 1 + 2(y - x)(-1)$$
$$\frac{\partial z}{\partial y} = 2(y - x).$$

Adding these, we get the desired differential equation. Thus, the surface defines a solution to the given equation and is, by definition, an integral surface.
Hildebrand, **Chapter 8, #9**:
Part (a). The Characteristic vector field is $V = (1, 1, 1)$, so the associated

ODE's are $dx = dy = dz$. We get that the characteristic curves are defined by $y - x = C_1$, $z - x = C_2$. This leads to the general solution

$$z - x = C_2 = f(C_1) = f(y - x)$$

where f is arbitrary.
Part (b). We must determine the function f which makes our solution consistent with the equations $z = y^2$ and $x = 0$. Plugging this information into our general form, we find

$$y^2 = z = x + f(y - x) = 0 + f(y + 0).$$

Therefore, our function must be $f(y) = y^2$. In this case, $x = x + (y - x)^2$, which is consistent with 8(c).
Hildebrand, **Chapter 8, #10**:
Part (a). We know that $z = x + f(y - x)$. Substitute in what we require to get

$$\phi(x) = z = x + f(2x - x) = x + f(x).$$

So that

$$f(x) = \phi(x) - x.$$

We deduce that

$$z = x + \phi(y - x) - (y - x) = 2x - y + \phi(y - x)$$

is our solution.
Part (b). Let's proceed by naively checking what the prescribed initial conditions mean. Along the curve $y = x$, we must have

$$\phi(x) = z = x + f(y - x) = x + f(0)$$

If we denote the constant $f(0)$ by k, we see that we must have $\phi(x) = x + k$. If ϕ does not have this form, then no solution is possible as our computation above shows. When ϕ is of this form, we can choose any function f for which $f(0) = k$ and specialize our general solution. Since there are infinitely many such functions, we see there are infinitely many solutions meeting this initial condition.

A remark: What has happened here is that the line $y = x$ in the xy-plane is the projection of a characteristic curve. So the situation is as we described in class.
Part (c). The characteristic vector field is $V = (1, 1, 1)$. We found in problem 8 that the characteristic curves for this equation are lines defined by

$$x - x_0 = y - y_0 = z - z_0.$$

These project onto the lines $x - x_0 = y - y_0$ in the xy-plane. Rearranging a bit, we see that the projections are $y = x + c$. Generalizing what we did in the last part, we see that to prescribe the value $\phi(x)$ along this line, we must have

$$\phi(x) = z = x + f(y - x) = x + f(c).$$

So, setting $k = f(c)$, we see that ϕ must be as described in the problem. Again, if ϕ meets this requirement, we can take an infinite number of different functions f (namely those with $f(c) = k$) as satisfying the initial condition ϕ along the curve $y = x + c$. Also, if ϕ does not meet this requirement, then there can be no solution.

Hildebrand, Chapter 8, # 11: Our equation is $z_x - z_y = 0$, so the characteristic vector field is $V = (1, -1, 0)$. The characteristic ODE's

$$dx = -dy = \frac{dz}{0}.$$

So we take the equations $dz = 0$ and $dy = -dx$. We solve these with $z = C_1$ and $y = -x + C_2$, respectively. Our general solution is thus

$$z = C_1 = f(C_2) = f(y + x).$$

Now, to find our particular solution, we substitute in what we know (in terms of t).

$$\begin{aligned}(t+1)^4 = z &= f(x+y) \\ &= f(t^2 + 1 + 2t) \\ &= f((t+1)^2).\end{aligned}$$

Hence, $f(a) = a^2$. Therefore, our solution is

$$z = (x+y)^2$$

Hildebrand, Chapter 8, # 12: Part (a). Our general solution is

$$z = \frac{c}{b}y + f(ay - bx)$$

We insert our desired initial data to find

$$x = \frac{c}{b}x + f(ax - bx).$$

Using the substitution $t = (a - b)x$, we see

$$f(t) = \frac{t}{a-b}\left(1 - \frac{c}{b}\right)$$

as long as $a - b \neq 0$. In this case, we get

$$z = \frac{c}{b}y + \frac{ay - bx}{a - b}\left(1 - \frac{c}{b}\right).$$

Now in the case when $a = b$, we instead find that $x = \frac{c}{b}x - f(0)$ is required. This is only possible when $c = b$ and $f(0) = 0$. If $c \neq b$, there is no particular solution meeting our prescription of initial condition. If $c = b$, then there are

infinitely many solutions meeting our prescription: one can choose any differentiable function f with $f(0) = 0$.
Part(b). The general solution we found was

$$z = e^{(cx/a + f(ay-bx))}.$$

Applying the same procedure as before, we see that

$$f((a-b)x) = \ln(x) - \left(\frac{c}{a}x\right).$$

Substituting $t = (a-b)x$, we get

$$f(t) = \ln\left(\frac{t}{a-b}\right) - \left(\frac{c}{a} \cdot \frac{t}{a-b}\right).$$

Which means that our solution is

$$\begin{aligned} z &= e^{(cx/a)+\ln\left(\frac{ay-bx}{a-b}\right)-\left(\frac{c}{a}\frac{ay-bx}{a-b}\right)}. \\ &= \frac{ay-bx}{a-b} \cdot e^{\left(\frac{c}{a-b}(y-x)\right)}. \end{aligned}$$

Again this only works when $a \neq b$. In the case where $a = b$, we obtain the necessary condition $x = \text{const} \cdot e^{cx/a}$. But this is impossible, so there is no solution meeting our conditions in this case.
Part (c). The general solution is $z = f(x^2+y^2)$. We insert our initial conditions to find that f must satisfy $x = f(2x^2)$. That is $f(x) = \sqrt{x/2}$. Thus, the particular solution we are after is

$$z = \sqrt{(x^2+y^2)/2}.$$

Part (d). The general solution is $z = e^{(-x^2+f(y-x))}$. We input the initial conditions to find

$$x = e^{-x^2} \cdot e^{f(0)}.$$

Again, as $f(0)$ is a constant, this is impossible. We conclude that there is no solution meeting our initial conditions.
Part (e). The general solution is $z = xf(xy)$. The initial conditions force that $x = xf(x^2)$, which means that $f(x) = 1$. So our particular solution is

$$z = x.$$

Part (f). Our general solution is

$$z = \frac{1}{\frac{1}{x} + f(\frac{1}{x} - \frac{1}{y})}.$$

When we put in the initial condition $x = y = z$ we find

$$x = \frac{1}{\frac{1}{x} + f(\frac{1}{x} - \frac{1}{x})}.$$

Rearranging this, we get the equation $f(0) = 0$. There are infinitely many solutions, we may use any differentiable function f for which $f(0) = 0$.
Hildebrand, **Chapter 8, # 17**:
Part (a). The C.V.F. is $V = (x, y, t, 1)$. The associated ODE's are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{t} = \frac{dz}{1}$$

We solve these with the equations $y = C_1 x$, $t = C_2 x$, and $z = C_3 + \ln x$. So the general solution is found by making C_3 a function of C_1 and C_2:

$$z = \ln x + f(y/x, t/x).$$

Part (b). *The solution in the book is incorrect!* To find the desired particular solution, we input all of the known information in a way that removes reference to the variables z and t.

$$\phi(x, y) = z = \ln x + f(y/x, t/x) = \ln x + f(y/x, (x^2 + y)/x).$$

We now set $u = y/x$ and $v = (x^2 + y)/x$. This means that $x = v - u$ and $y = (v - u)u$. Using this substitution, we find

$$\phi(v - u, (v - u)u) = \ln(v - u) + f(u, v).$$

If we use this definition of f, we get the particular solution

$$\begin{aligned} z &= \ln x + \phi\left(\frac{t}{x} - \frac{y}{x}, \left(\frac{t}{x} - \frac{y}{x}\right)\frac{y}{x}\right) - \ln\left(\frac{t}{x} - \frac{y}{x}\right) \\ &= \phi\left(\frac{t - y}{x}, \frac{(t - y)y}{x^2}\right) + \ln\left(\frac{x^2}{t - y}\right). \end{aligned}$$