Homework Assignment # 2
Due Friday, September 19, 2003.

1. Calculate the coefficients in the Fourier Series for a function $f(x)$ of period 2π which is equal to -1 for $-\pi < x < 0$ and equal to 1 for $0 < x < \pi$.

2. Calculate the coefficients in the Fourier Series for an even function of period 2π which is equal to x for $0 \leq x \leq \pi/2$ and equal to $\pi/2$ for $\pi/2 \leq x < \pi$.

3. Let $f(x) = \frac{1}{2}(\pi - x)$. Find the Fourier series of the extension of f to an odd function of period 2π.

4. Construct a graph of the first three partial sums of the Fourier series of the functions in problems 1, 2 and 3 on the same set of axes as a graph of the original function.

5. Let $s_n(x)$ denote the n^{th} order partial sum of the Fourier series for a function f. Show that

$$\int_{-\pi}^{\pi} s_n(x)^2 dx = \pi \frac{a_0^2}{2} + \pi \sum_{k=1}^{n} (a_k^2 + b_k^2),$$

where the a_k's and b_k's are the Fourier coefficients of f.

6. Show that

$$\int_{-\pi}^{\pi} \frac{\sin\left((n+\frac{1}{2})u\right)}{2\sin(\frac{u}{2})} du = \pi.$$

(Hint: Recall the original definition of the function $G(v)$ from class.)

7. Let f be a 2π-periodic function for which $f(x) = x$ when $-\pi \leq x < \pi$. By considering the Fourier series of f, write an infinite sum expansion of π in the form

$$\pi = 4 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots).$$

What about the function f prevents us from just evaluating the Fourier series at the point $x = \pi$?

8. By a similar method using the 2π-periodic function for which $g(x) = |x|$ when $-\pi \leq x \leq \pi$, find an infinite sum expansion for π^2. *(Hint: look at the point $x = 0$.)* How big of a partial sum must one use to ensure the error in using this series to estimate π is less than 10^{-3}? *(Hint: Look at the proof of Theorem 5.)* What about the function f might make the convergence so slow?

Problems 9 and 10 collect facts that we need for the lecture on the Gibbs phenomenon.

9. Let ψ be the function considered in problem 3. Use the trig identity

$$\cos(t) + \cdots + \cos(nt) = -\frac{1}{2} + \frac{\sin\left((n+\frac{1}{2})t\right)}{2\sin(\frac{t}{2})}$$

to show that the n^{th} partial sum of ψ at $x \in (0, 2\pi)$ is

$$s_n(x) = -\frac{x}{2} + \int_0^x \frac{\sin\left((n+\frac{1}{2})u\right)}{2\sin(\frac{u}{2})} du.$$

10. Continuing from 9, let $R_n(x) = \psi(x) - s_n(x)$ be the n^{th} remainder. Show that the derivative of $R_n(x)$ is

$$R_n'(x) = -\frac{\sin\left((n+\frac{1}{2})x\right)}{2\sin(\frac{x}{2})}.$$

Show that in the interval $(0, 2\pi)$, R_n' has zeroes exactly at the points

$$x_1 = \frac{2\pi}{2n+1}, \quad x_2 = \frac{2\cdot 2\pi}{2n+1}, \ldots, \quad x_{2n} = \frac{2n \cdot 2\pi}{2n+1}.$$

Which of these are maxima and which are minima of R_n? Show that at a minimum point $R_n(x_i) < 0$ and at a maximum point $R_n(x_i) > 0$. Observe that this means that each of the points x_i is a maximum point for the function $|R_n(x)| = |\psi(x) - s_n(x)|$.