Solutions to Homework # 2
Math 381, Rice University, Fall 2003

Problem1: The function is odd, so all of the a_k's are zero, including a_0. We then compute the b_k's to be

$$\begin{aligned} b_k &= \frac{2}{\pi}\int_0^{\pi} \sin(kx)dx = \frac{-2}{k\pi}\left(\cos(k\pi)-\cos(0)\right) = \frac{-2}{k\pi}\left((-1)^k-1\right) \\ &= \begin{cases} 0, & k \text{ even} \\ \frac{4}{k\pi}, & k \text{ odd} \end{cases}. \end{aligned}$$

The resulting Fourier series is

$$f(x) = \frac{4}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \dots\right)$$

Problem 2: This function is even, so all of the b_k's are zero. The average value of f is

$$A_0 = \frac{a_0}{2} = \frac{2}{2\pi}\left(\int_0^{\pi/2} x\ dx + \int_{\pi/2}^{\pi} \frac{\pi}{2}\ dx\right) = \frac{3\pi}{8}$$

To get the other a_k's, we compute that

$$\begin{aligned} a_k &= \frac{2}{\pi}\left(\int_0^{\pi/2} x\cos(kx)\ dx + \int_{\pi/2}^{\pi} \frac{\pi}{2}\cos(kx)\ dx\right) \\ &= \frac{1}{k}\sin(\frac{k\pi}{2}) + \frac{2}{\pi k^2}\left(\cos(\frac{k\pi}{2})\right) - \frac{1}{k}\sin(\frac{k\pi}{2}) \\ &= \begin{cases} -\frac{2}{\pi k^2}, & k \text{ odd} \\ 0, & k \text{ is divisible by 4} \\ -\frac{4}{\pi k^2}, & k \text{ is divisible by 2 but not 4} \end{cases} \end{aligned}$$

This is a bit hard to see at first, maybe. But the first few terms written down are

$$\begin{aligned} f(x) &= \frac{3\pi}{8} + \sum_{k=1}^{\infty} \frac{2}{\pi k^2}\left(\cos(\frac{k\pi}{2}) - 1\right)\cos(kx) \\ &= \frac{3\pi}{8} - \frac{2}{\pi}\left(\cos(x) + \frac{1}{3^2}\cos(3x) + \dots\right) - \frac{4}{\pi}\left(\frac{1}{2^2}\cos(2x) + \frac{1}{6^2}\cos(6x) + \dots\right) \end{aligned}$$

Problem 3: The function is odd, so we know that all of the a_k's are zero. We then compute that the b_k's are (actual work omitted, it's more integration by parts. . .) $b_k = \frac{1}{k}$. Because we'll need it for later problems, we record that the resulting Fourier series looks like

$$\psi(x) = \sin(x) + \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} + \dots$$

Problem 4: The answer here is a bunch of pretty pictures. I'll have to attach these later.
Problem 5: First consider the square of the partial sum

$$s_n(x)^2 = \left(\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx)) \right)^2.$$

When you multiply this out term-by-term, we get the term $\frac{a_0^2}{4}$ and four other kinds of terms in the answer.

- terms like $a_0 a_k \cos(kx)$ or $a_0 b_k \sin(kx)$
- terms like $a_p \cos(px) \cdot b_q \sin(qx)$
- terms like $a_p \cos(px) \cdot a_q \cos(qx)$
- terms like $b_p \sin(px) \cdot b_q \sin(qx)$

The trick is to notice that the next step is to integrate this from $-\pi$ to π. Using the facts about integrals of sines and cosines from the first day of lecture on Fourier series, we see that all of the terms of the first two types integrate out to zero and the same happens in the second two types unless $p = q$. Applying this, we see

$$\begin{aligned}\int_{-\pi}^{\pi} s_n(x)^2 dx &= \int_{-\pi}^{\pi} \left(\frac{a_0^2}{4} + \sum_{k=1}^{n} (a_k^2 \cos^2(kx) + b_k^2 \sin^2(kx)) \right) dx \\ &= \pi \frac{a_0^2}{2} + \pi \sum_{k=1}^{n} (a_k^2 + b_k^2).\end{aligned}$$

This completes the exercise.
Problem 6: Recall the trig identity from class:

$$\frac{1}{2} + \cos(t) + \cdots + \cos(nt) = \frac{\sin\left((n + \frac{1}{2})t\right)}{2\sin(\frac{t}{2})}.$$

If you integrate this term-by-term from $-\pi$ to π, all of the integrals on the left vanish except for the first one.

$$\int_{-\pi}^{\pi} \frac{1}{2} dt = \int_{-\pi}^{\pi} \frac{\sin\left((n + \frac{1}{2})t\right)}{2\sin(\frac{t}{2})} dt.$$

Since the integral on the left side of this equation is equal to π, we are done.
Problem 7: The function quoted in the problem has a Fourier series which converges at $\pi/2$. Using this fact we see that

$$\begin{aligned}\frac{\pi}{2} &= 2\left(\sin(\frac{\pi}{2}) - \frac{1}{2}\sin(2\frac{\pi}{2}) + \frac{1}{3}\sin(3\frac{\pi}{2}) - \frac{1}{4}\sin(4\frac{\pi}{2}) + \dots \right) \\ &= 2(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots).\end{aligned}$$

We then multiply this through by 2 by get the desired result. (This operation on the infinite series is justified because the series converges.)

We cannot do this by plugging in the point π because the Fourier series does not converge at π.

Problem 8: We take the same approach as in the last problem and (following the hint) evaluate the Fourier series of the given function at $x = 0$. This series converges at this point, so we learn that

$$\begin{aligned} 0 &= \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(0) + \frac{1}{3^2}\cos(0) + \frac{1}{5^2}\cos(0) + \frac{1}{7^2}\cos(0) + \dots\right) \\ &= \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}. \end{aligned}$$

Again, the series converges, so we can rearrange things to read

$$\pi^2 = 8\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = 8\left(1 + \frac{1}{3} + \frac{1}{25} + \dots\right).$$

Now, to estimate the error, we use the fact that this function is a broken line function. So by Theorem 5, the difference between $f(0)$ and $s_n(0)$ is no bigger than $\frac{2C}{n}$, where from the proof of Theorem 2, $C = \frac{2m\lambda}{\pi}$, λ is the maximum of the absolute values of the slopes in the function and m is the number of line segments used in a single period. For us, $\lambda = 1$ and $m = 2$. Thus, we need to choose n so large that

$$|f(0) - s_n(0)| \leq \frac{2C}{n} = \frac{8}{\pi n} < \frac{1}{1000}.$$

This means that $n > \frac{8000}{\pi} > 2547$ will do. The convergence is so slow because this function has a point where the derivative is discontinuous. (In fact, we are looking at that particular point!)

Problem 9: We begin by integrating the quoted trig identity from 0 to x.

$$\begin{aligned} \int_0^x (\cos(t) + \cdots + \cos(nt))dt &= \int_0^x \left(-\frac{1}{2} + \frac{\sin\left((n+\frac{1}{2})t\right)}{2\sin(\frac{t}{2})}\right) dt \\ &= -\frac{x}{2} + \int_0^x \left(\frac{\sin\left((n+\frac{1}{2})t\right)}{2\sin(\frac{t}{2})}\right) dt. \end{aligned}$$

But the left hand side of this is equal to

$$\sin(x) + \frac{1}{2}\sin(2x) + \cdots + \frac{1}{n}\sin(nx) = s_n(x),$$

so we are done.

Problem 10: Since $R_n(x) = \psi(x) - s_n(x)$, we use the formula from problem 9 to see that

$$\begin{aligned} R_n(x) &= \frac{1}{2}(\pi - x) - \left(-\frac{x}{2} + \int_0^x \left(\frac{\sin\left((n+\frac{1}{2})t\right)}{2\sin(\frac{t}{2})} \right) dt \right) \\ &= \frac{\pi}{2} - \int_0^x \left(\frac{\sin\left((n+\frac{1}{2})t\right)}{2\sin(\frac{t}{2})} \right) dt. \end{aligned}$$

So by the Fundamental Theorem of Calculus, we get

$$R_n'(x) = -\frac{\sin\left((n+\frac{1}{2})x\right)}{2\sin(\frac{x}{2})}.$$

Next, we look for the zeros of this function in the interval $(0, 2\pi)$. The denominator doesn't vanish, so we only have to check when the numerator does vanish to find the zeros of R_n'. But $\sin((n+\frac{1}{2})x) = 0$ precisely when $(n+\frac{1}{2})x$ is a multiple of π. So we get that the zeros have the form $x_k = \frac{\pi \cdot k}{(n+\frac{1}{2})} = \frac{2\pi \cdot k}{2n+1}$. When you check which ones lie in the interval $(0, 2\pi)$, you get the ones described in the problem, namely, k runs from 1 to $2n$.

To check which of these give maxima and which give minima of R_n, we use the sign test on the first derivative. By checking the points $\frac{\pi}{2n+1}, \frac{3\pi}{2n+1}, \ldots, \frac{(2n-1)\pi}{2n+1}$, we see that the signs alternate from negative positive to negative to positive to negative to positive to... This means that x_k for k odd is a minimum, and x_k for k even is a maximum. Another way is to use the second derivative test. Any valid method will be accepted.

The easiest way to check the claim about $R_n(x)$ being positive at maxima and negative at minima is to use numerical approximations to the integrals you get. This is good enough for me: you can quote MatLab or Mathematica or Maple or your scientific calculator. Any valid method will be accepted.

Now the observation to be made is that for $|R_n(x)|$ we get local maxima for all of the points above because the absolute value will reflect the negative parts of the graph up. Of course, the critical points of $|R_n(x)|$ are exactly the points which are either critical or a zero for R_n, but the zeros turn into the minima of $|R_n(x)|$.