Solutions to Homework # 3
Math 381, Rice University, Fall 2003

Problem 1: (*Hildebrand*, Chapter 9 #15) This problem is a lot like the one we described in lecture on Monday, September 22. The only difference is in the boundary conditions.

Since the equation is homogeneous, we can simply add the solutions to the two boundary value problems we get by setting f or g to be exactly zero.

The problem with $g(x) = 0$ we solved in class with

$$u_1(x, y) = \sum_{k=1}^{\infty} c_k \sinh\left(\frac{k\pi}{l}(d-y)\right) \sin\left(\frac{k\pi}{l}x\right)$$

If you work the details out for problem with the other boundary conditions (that is, set $f = 0$), we get

$$u_2(x, y) = \sum_{k=1}^{\infty} d_k \sinh\left(\frac{k\pi}{l}y\right) \sin\left(\frac{k\pi}{l}x\right)$$

as a solution. Adding these two, we still get a solution to Laplace's equation. It is not to hard to check that the sum then meets the required boundary conditions. So our formal solution is

$$u(x, y) = \sum_{k=1}^{\infty} \left(c_k \sinh\left(\frac{k\pi}{l}(d-y)\right) + d_k \sinh\left(\frac{k\pi}{l}y\right)\right) \sin\left(\frac{k\pi}{l}x\right),$$

where the coefficients c_k and d_k are chosen so that

$$c_k \sinh\left(\frac{k\pi}{l}(d)\right)$$

is the k^{th} coefficient in the Fourier sine series for $f(x)$ and

$$d_k \sinh\left(\frac{k\pi}{l}(d)\right)$$

is the k^{th} coefficient in the Fourier sine series for $g(x)$.

To insure that this formal solution is an actual solution, one needs to check convergence of the series. From the form of the sum, it is sufficient that the functions f and g are nice enough to have convergent Fourier Series.

Problem 2: (*Hildebrand*, Chapter 9 #21) The idea here is that a polynomial is identically zero only if all its coefficients are zero. So, suppose we have a polynomial in x and y. As I suggested in email, it is enough to consider something cubic (that's enough to do problem 3). So, consider

$$\varphi = A_0 + A_1x + A_2y + A_3x^2 + A_4xy + A_5y^2 + A_6x^3 + A_7x^2y + A_8xy^2 + A_9y^3.$$

We compute the first few derivatives to get

$$\begin{aligned}\frac{\partial^2\varphi}{\partial x^2} &= 2A_3 + 6A_6x + 2A_7y \\ \frac{\partial^2\varphi}{\partial y^2} &= 2A_5 + 2A_8x + 6A_9y.\end{aligned}$$

So that

$$\begin{aligned}0 = \triangle\varphi &= \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} \\ &= (2A_3 + 6A_6x + 2A_7y) + (2A_5 + 2A_8x + 6A_9y) \\ &= 2(A_3 + A_5) + 2x(3A_6 + A_8) + 3y(A_7 + 3A_9).\end{aligned}$$

So, we see that there are no restrictions on A_0, A_1, A_2, A_4, and that $A_5 = -A_3$, $A_8 = -3A_6$ and $A_7 = -3A_9$. This means that φ is harmonic when it takes the form

$$\varphi = A_0 + A_1x + A_2y + A_4xy + A_3(x^2 - y^2) + A_6(x^3 - 3xy^2) + A_9(y^3 - 3x^2y).$$

Relabelling the constants, we get the desired result

$$\varphi = a_0 + a_1x + a_2y + a_3xy + a_4(x^2 - y^2) + a_5(x^3 - 3xy^2) + a_6(y^3 - 3x^2y).$$

Problem 3: (*Hildebrand*, Chapter 9 #22) Following the hint in the text, we split our problem into two different boundary value problems. Namely, set $T = p(x, y) + u(x, y)$ where $p(x, y)$ is a polynomial (and hence grows slower than any exponential). Then we have two problems:

Problem One

$$\begin{cases} \triangle p(x,y) = 0, & \text{for } 0 \le y \le \infty \text{ and } 0 \le x \le l, \\ u(0,y) = \alpha_1 + \beta_2 y, & \text{for all } y, \\ u(l,y) = \alpha_2 + \beta_2 y, & \text{for all } y, \\ u(x,0) = 0, & \text{for all } x, \\ p \text{ is a polynomial.} & \end{cases}$$

Problem Two

$$\begin{cases} \triangle u(x,y) = 0, & \text{for } 0 \le y \le \infty \text{ and } 0 \le x \le l, \\ u(0,y) = u(l,y) = 0, & \text{for all } y, \\ u(x,0) = f(x) - p(x,0), & \text{for all } x, \\ \lim_{y\to\infty} u(x,y) \to 0, & \text{for all } x \end{cases}$$

The sum of the solutions to these problems will give us a solution to the stated problem. Note that we have adjusted the boundary value for problem two in terms of what problem one gives us, that is because we can't set the corresponding boundary condition to zero in our polynomial problem and still get a (nontrivial) solution.

To solve the first problem, we use the Problem 2. We see that a polynomial solution to Laplace's equation has the form

$$p(x,y) = a_0 + a_1 x + a_2 y + a_3 xy + a_4(x^2 - y^2) + a_5(x^3 - 3xy^2) + a_6(y^3 - 3x^2 y) + \dots$$

To meet the boundary conditions, we see

$$\alpha_1 + \beta_1 y = p(0,y) = a_0 + a_2 y - a_4 y^2 + \text{higher order terms in } y \dots$$

So that $a_0 = \alpha_1$ and $a_2 = \beta_1$. All of the other terms which appear here (from a_4 on up) must vanish. This means that we only need to look at

$$p(x,y) = \alpha_1 + a_1 x + \beta_1 y + a_3 xy.$$

We have one more initial condition to check, it is

$$\alpha_2 + \beta_2 y = p(l,y) = \alpha_1 + a_1 l + \beta_1 y + a_3 ly.$$

Comparing like terms, we solve for the coefficients $a_1 = \frac{\alpha_2 - \alpha_1}{l}$ and $a_3 = \frac{\beta_2 - \beta_1}{l}$. This fixes up the solution to problem one as

$$p(x,y) = \alpha_1 + \frac{\alpha_2 - \alpha_1}{l} x + \beta_1 y + \frac{\beta_2 - \beta_1}{l} xy$$

We have already solved problem two. It is the first example from class. The solution is

$$u(x,y) = \sum_{k=1}^{\infty} b_k e^{-\frac{k\pi y}{l}} \sin\left(\frac{k\pi x}{l}\right),$$

where the b_k's are the Fourier Sine coefficients of $f(x) - p(x,0) = f(x) - \alpha_1 - (\alpha_2 - \alpha_1)(x/l)$. Note that this does not grow exponentially, it dies exponentially.

So our final solution is

$$T(x,y) = \alpha_1 + \frac{\alpha_2 - \alpha_1}{l} x + \beta_1 y + \frac{\beta_2 - \beta_1}{l} xy + \sum_{k=1}^{\infty} b_k e^{-\frac{k\pi y}{l}} \sin\left(\frac{k\pi x}{l}\right),$$

where the b_k's are the Fourier Sine series coefficients of

$$g(x) = f(x) - \alpha_1 - (\alpha_2 - \alpha_1)(x/l).$$

Problem 4: For this problem, we will use the solution to the plucked string wave equation that we derived in class. We need some nice initial condition, so we use

$$f(x) = \begin{cases} \frac{h}{b}x, & 0 \le x \le b \\ \frac{h}{\pi - b}(\pi - x) & b \le x \le \pi. \end{cases}$$

This describes a string which is pulled above horizontal to a height h over the point $b \in (0, \pi)$.

The solution we found in class to the plucked string problem requires the Fourier Sine series coefficients of f. So we compute these to be

$$b_k = \ldots = \frac{2h}{k^2} \frac{\sin(kb)}{b(\pi - b)}.$$

Recall our discussion of how the sound only depends on the relative strengths of those frequencies which appear in this Fourier expansion.

Now we can readily answer questions (a) and (b). First, for (b), changing how hard you pluck the string corresponds to changing the height h. We see that a change in h affects all of the Fourier coefficients in the same way, so all we change is the volume of the sound. The relative strengths of the vibration in every frequency is affected in the same way, so the note does not change. But the total energy of the waves changes, so the volume is different. As for (a), it does matter where you pluck a string. changing the value of b affects the Fourier coefficients differently depending on the value of k. Therefore, the relative strengths of the frequencies you hear is changed, and the string will not sound the same. As an example, if $b = \pi/2$ then the second harmonic is missing but the third is present. But if $b = \pi/3$, the third harmonic is missing but the second is present. Note however that in this case the total energy is preserved (why is that? look at the geometry of the integral), so the volume is left unchanged.

To determine the answer to (c), we need to solve the wave equation with a different set of initial conditions. The resulting solution is

$$y(x,t) = \sum_{k=1}^{\infty} c_k \sin(kx)\sin(kat)$$

where

$$f(x) = y_t(x,0) = \sum_{k=1}^{\infty} kac_k \sin(kx).$$

So that kac_k is the Fourier Sine series coefficient of f. This means that at the same frequency, the struck string has energy equal to $c_k = \frac{b_k}{ka}$, which is different from the energy b_k of the k^{th} harmonic frequency of the plucked string. So it *matters.*

Of course, it is hard to "strike" a string and give it initial velocities determined by the function f above. In reality, you still "pull and release" a string when you strum it... Which is fortunate, because that gives a richer sound.