Homework Assignment # 4
Due Friday, October 17, 2003.

Problem 1: Solve the general Euler equation

$$x^2y'' + \alpha xy' + \beta y = 0$$

where α and β are constants. (Hint: generalize what we did in class.)

Problem 2: Show the work required to verify that in cylindrical coordinates (r, θ, z), the Laplace operator takes the form

$$\triangle u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$

(Recall that cylindrical coordinates are related to rectangular coordinates (x, y, z) by $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.)

Problem 3: Show the work required to verify that in spherical coordinates (r, θ, φ), the Laplace equation takes the form

$$\triangle u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}.$$

(Recall that spherical coordinates are related to rectangular coordinates (x, y, z) by $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$.)

Problem 4: The flow of heat in an insulated wire ring is described by the heat equation with periodic boundary conditions. If $u(t, \theta)$ is the temperature of the ring at time t and angular coordinate θ, and the initial temperature distribution is given by $f(\theta)$, then we get the following conditions.

$$\begin{cases} \dfrac{1}{\alpha^2}\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial \theta^2}, & 0 \le \theta \le 2\pi, 0 \le t < \infty \\ u(t, 0) = u(t, 2\pi), & 0 \le t < \infty \\ u(0, \theta) = f(\theta), & 0 \le \theta \le 2\pi. \end{cases}$$

Here, α^2 is just some physical constant relating to the specific heat of the wire. Solve these equations using separation of variables to find a description for the flow of heat in the wire ring. What is the limiting behavior of this function, and what does it mean for the "final" temperature? Does this make physical sense?

Problem 5: This problem has three parts.

a) Using rectangular coordinates, show that the function

$$u(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$$

is harmonic away from the origin. (The origin is a bad point for this function.)

b) In spherical coordinates, this function takes the form

$$u(r, \theta, \varphi) = \frac{1}{r}.$$

Using spherical coordinates, show that this function is harmonic.

c) Going back to rectangular coordinates, show that if u is a harmonic function and the point (x_0, y_0, z_0) is fixed, then the translated function $v(x, y, z) = u(x - x_0, y - y_0, z - z_0)$ is also harmonic.

Problem 6: Using the recurrence formula for Legendre polynomials and the fact that $P_0(x) = 1$ and $P_1(x) = x$, compute the Legendre polynomials up to $P_7(x)$. Draw graphs of these eight functions P_0 through P_7 on the interval $(-1, 1)$. (It would be preferable to have all the graphs on one set of axes.)