Solutions to Homework # 4
Math 381, Rice University, Fall 2003

Problem 1: We start by making the Euler substitution $t = \ln x$, or $x = e^t$. Then we compute that

$$\frac{dy}{dt} = xy', \quad \text{and} \quad \frac{d^2y}{dt^2} = x^2y'' + xy'.$$

So we can rewrite our original equation as

$$\frac{d^2y}{dt^2} + (\alpha - 1)\frac{dy}{dt} + \beta y = 0. \tag{1}$$

This is now a constant coefficient, homogeneous, linear second order ODE. The theory of these equations tells us to consider the characteristic polynomial

$$\lambda^2 + (\alpha - 1)\lambda + \beta = 0 \tag{2}$$

of this equation, and that the solutions of equation (1) depend on the nature of the roots to this characteristic polynomial. By the quadratic formula, we get that the roots are

$$\lambda_{1,2} = \frac{1 - \alpha \pm \sqrt{(\alpha - 1)^2 - 4\beta}}{2}.$$

The solutions to (1) fall into three cases depending on the sign of the *discriminant* $D = (\alpha - 1)^2 - 4\beta$.

Case One: ***D is positive*** In this case, there are two distinct real roots. The general solution to (1) is then

$$y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}.$$

Changing variables back to x, we get

$$y(x) = Ax^{\lambda_1} + Bx^{\lambda_2},$$

where λ_1, λ_2 are the roots of (2).

Case Two: $D = 0$ In this case, (2) has one real root and it is a double root. The general solution to (1) is then given by

$$y(t) = Ae^{\lambda_1 t} + Bte^{\lambda_2 t}.$$

Unwinding the Euler substitution, we get that our solution is

$$y(x) = Ax^{\lambda} + Bx^{\lambda}\ln(x),$$

where λ is the root of (2).

Case Three: D ***is negative*** In this case, (2) has a pair of conjugate complex roots. These will be

$$\lambda = \frac{1-\alpha}{2} + \imath\frac{\sqrt{4\beta - (\alpha-1)^2}}{2},$$

and

$$\overline{\lambda} = \frac{1-\alpha}{2} - \imath\frac{\sqrt{4\beta - (\alpha-1)^2}}{2}.$$

The solution of (1) is given by

$$y(t) = e^{\frac{1-\alpha}{2}t}\left(A\cos\left(\frac{\sqrt{4\beta-(\alpha-1)^2}}{2}t\right) + B\sin\left(\frac{\sqrt{4\beta-(\alpha-1)^2}}{2}t\right)\right).$$

Translating things back through the Euler substitution, we get

$$y(x) = x^{\frac{1-\alpha}{2}}\left(A\cos\left(\frac{\sqrt{4\beta-(\alpha-1)^2}}{2}\ln(x)\right) + B\sin\left(\frac{\sqrt{4\beta-(\alpha-1)^2}}{2}\ln(x)\right)\right).$$

Note that though this is very descriptive, it is ugly to write in general. That is why it is better to memorize the technique than the answer.

Problem 2: (For this problem and the next, we use the subscript notation for derivatives.) We follow the procedure we outlined in class for the case of polar coordinates in the plane. We find first that

$$\begin{aligned}
u_r =& \cos\theta\ u_x + \sin\theta\ u_y \\
u_\theta =& -r\sin\theta\ u_x + r\cos\theta\ u_y \\
u_z =& u_z.
\end{aligned}$$

We then solve these equations for u_x, u_y and u_z to find

$$\begin{aligned}
u_x &= \cos\theta\ u_r - \frac{1}{r}\sin\theta\ u_\theta \\
u_y &= \sin\theta\ u_r + \frac{1}{r}\cos\theta\ u_\theta \\
u_z &= u_z.
\end{aligned}$$

Now, we use these equations to compute out what the terms of the Laplace operator look like in our new coordinates.

$$\begin{aligned}
u_{xx} &= \frac{\partial}{\partial x}(u_x) = \cos\theta(u_x)_r - \frac{1}{r}\sin\theta(u_x)_\theta \\
&= \cos\theta\left[\cos\theta\ u_{rr} + \frac{1}{r^2}\sin\theta\ u_\theta - \frac{1}{r}\sin\theta\ u_{r\theta}\right] \\
&\quad - \frac{1}{r}\sin\theta\left[-\sin\theta\ u_r + \cos\theta\ u_{r\theta} - \frac{1}{r}\cos\theta\ u_\theta - \frac{1}{r}\sin\theta\ u_{\theta\theta}\right]
\end{aligned}$$

Similarly,

$$\begin{aligned}
u_{yy} &= \frac{\partial}{\partial y}(u_y) = \sin\theta(u_y)_r + \frac{1}{r}\cos\theta(u_y)_\theta \\
&= \sin\theta\left[\sin\theta\ u_{rr} - \frac{1}{r^2}\cos\theta\ u_\theta + \frac{1}{r}\cos\theta\ u_{r\theta}\right] \\
&\quad + \frac{1}{r}\cos\theta\left[\cos\theta\ u_r + \sin\theta\ u_{r\theta} - \frac{1}{r}\sin\theta\ u_\theta - \frac{1}{r}\cos\theta\ u_{\theta\theta}\right]
\end{aligned}$$

Now, we combine these to find that

$$\triangle u = u_{xx} + u_{yy} + u_{zz} = u_{rr} + \frac{1}{r}\ u_r + \frac{1}{r^2}\ u_{\theta\theta} + u_{zz}$$

Problem 3: The procedure is the same for this problem as the last one. We find the change of coordinate formulae for spherical derivatives in terms of rectangular ones, invert this system of linear equations to find rectangular derivatives in terms of spherical ones, and then compute out what the terms of the Laplace operator look like in spherical coordinates and add.

So, we begin with

$$\begin{aligned}
u_r &= \sin\theta\cos\varphi\ u_x + \sin\theta\sin\varphi\ u_y + \cos\theta\ u_z \\
u_\theta &= r\cos\theta\cos\varphi\ u_x + r\cos\theta\sin\varphi\ u_y - r\sin\theta\ u_z \\
u_\varphi &= -r\sin\theta\sin\varphi\ u_x + r\sin\theta\cos\varphi\ u_y.
\end{aligned}$$

When you invert this relationship (it helps to use Matlab. . .) you find

$$\begin{aligned}
u_x &= \sin\theta\cos\varphi\ u_r + \frac{1}{r}\cos\theta\cos\varphi\ u_\theta - \frac{\sin\varphi}{r\sin\theta}\ u_\varphi \\
u_y &= \sin\theta\sin\varphi\ u_r + \frac{1}{r}\cos\theta\sin\varphi\ u_\theta + \frac{\cos\varphi}{r\sin\theta}\ u_\varphi \\
u_z &= \cos\theta\ u_x - \frac{1}{r}\sin\theta\ u_\theta.
\end{aligned}$$

Now for the tricky bit. We take each term of the Laplace operator and write it in terms of the spherical coordinates. First,

$$\begin{aligned}
u_{xx} = \sin\theta\cos\varphi &\left[\sin\theta\cos\varphi\ u_{rr} - \frac{1}{r^2}\cos\theta\cos\varphi\ u_\theta + \frac{1}{r}\cos\theta\cos\varphi\ u_{r\theta} + \frac{\sin\varphi}{r^2\sin\theta}\ u_\varphi - \frac{\sin\varphi}{r\sin\theta}\ u_{r\varphi}\right] \\
&+ \frac{1}{r}\cos\theta\cos\varphi\left[\cos\theta\cos\varphi\ u_r + \sin\theta\cos\varphi\ u_{r\theta} - \frac{1}{r}\sin\theta\cos\phi\ u_\theta + \frac{1}{r}\cos\theta\cos\phi\ u_{\theta\theta}\right. \\
&\qquad\qquad\left. - \frac{\sin\varphi}{r\sin\theta}\ u_{\varphi\theta} + \frac{\cos\theta\sin\varphi}{r\sin^2\theta}\ u_\varphi\right] \\
&- \frac{\sin\varphi}{r\sin\theta}\left[-\sin\theta\sin\varphi\ u_r + \sin\theta\cos\varphi\ u_{r\varphi} + \frac{1}{r}\cos\theta\cos\varphi\ u_{\varphi\theta} - \frac{1}{r}\cos\theta\sin\varphi\ u_\theta\right. \\
&\qquad\qquad\left. - \frac{\cos\varphi}{r\sin\theta}\ u_\varphi - \frac{\sin\varphi}{r\sin\theta}\ u_{\varphi\varphi}\right].
\end{aligned}$$

And then,

$$\begin{aligned}
u_{yy} = \sin\theta\sin\varphi &\left[\sin\theta\sin\varphi\ u_{rr} - \frac{1}{r^2}\cos\theta\sin\varphi\ u_\theta + \frac{1}{r}\cos\theta\sin\varphi\ u_{r\theta} - \frac{\cos\varphi}{r^2\sin\theta}\ u_\varphi + \frac{\cos\varphi}{r\sin\theta}\ u_{r\varphi}\right] \\
&+ \frac{1}{r}\cos\theta\sin\varphi\left[\cos\theta\sin\varphi\ u_r + \sin\theta\sin\varphi\ u_{r\theta} - \frac{1}{r}\sin\theta\sin\phi\ u_\theta + \frac{1}{r}\cos\theta\sin\phi\ u_{\theta\theta}\right. \\
&\qquad\qquad\left. + \frac{\cos\varphi}{r\sin\theta}\ u_{\varphi\theta} - \frac{\cos\theta\cos\varphi}{r\sin^2\theta}\ u_\varphi\right] \\
&+ \frac{\cos\varphi}{r\sin\theta}\left[\sin\theta\cos\varphi\ u_r + \sin\theta\sin\varphi\ u_{r\varphi} + \frac{1}{r}\cos\theta\sin\varphi\ u_{\varphi\theta} + \frac{1}{r}\cos\theta\cos\varphi\ u_\theta\right. \\
&\qquad\qquad\left. - \frac{\sin\varphi}{r\sin\theta}\ u_\varphi + \frac{\cos\varphi}{r\sin\theta}\ u_{\varphi\varphi}\right].
\end{aligned}$$

Before things get any worse, lets add these. Lots of stuff cancels, and we use the trig identity $\cos^2\alpha + \sin^2\alpha = 1$ about 10 times to find

$$\begin{aligned} u_{xx} + u_{yy} = \sin^2\theta\ u_{rr} &+ \left[-\frac{2}{r^2}\cos\theta\sin\theta + \frac{\cos\theta}{r^2\sin\theta}\right]\ u_\theta + \frac{2\sin\theta\cos\theta}{r}\ u_{r\theta} \\ &+ \left[\frac{1}{r} + \frac{\cos^2\theta}{r}\right]\ u_r + \frac{\cos^2\theta}{r^2}\ u_{\theta\theta} + \frac{1}{r^2\sin^2\theta}\ u_{\varphi\varphi} \end{aligned}$$

Finally, we find that

$$u_{zz} = \cos^2\theta\ u_{rr} + \frac{\sin^2\theta}{r}\ u_r + \frac{2\sin\theta\cos\theta}{r^2}\ u_\theta - \frac{2\sin\theta\cos\theta}{r}\ u_{r\theta} + \frac{\sin^2\theta}{r^2}\ u_{\theta\theta}.$$

Adding these last two equations, we find that

$$\begin{aligned} \triangle u &= u_{xx} + u_{yy} + u_{zz} \\ &= u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta}\ u_\theta + u_{\theta\theta}\right) + \frac{1}{r^2\sin^2\theta}\ u_{\varphi\varphi} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}. \end{aligned}$$

So everything works out fine.

Problem 4: We separate the variables by assuming that u has the form $u(t,\theta) = T(t) \cdot S(\theta)$. We put this into the differential equation to find

$$\frac{1}{\alpha^2} T' S = T S'',$$

which means that

$$\frac{T'}{\alpha^2 T} = \frac{S''}{S}.$$

The left hand side of this equation is independent of θ and the right hand side is independent of t, so this quantity is a constant. Call it C. Then we get a pair of ODE's to solve.

$$S''(\theta) = C \cdot S(\theta)$$
$$T'(t) = \alpha^2 C \cdot T(t)$$

We know that to solve the first one we have two cases. If C is positive, we get exponentials. If C is negative, we get trig functions. In order to meet the boundary conditions, we want S to be 2π-periodic, so we take $C = -\lambda^2$ to be negative. (To see this, note that periodicity means that $T(t) = T(t+2\pi)$ for all t. If you write down what this means for the choices of constants C, D and k in a possible solution $T(t) = Ce^{kt} + De^{-kt}$, you can solve the equation for t. In that result, the one side is a constant, but t is arbitrary. This is a contradiction.)

The solution is then

$$S(\theta) = A\sin(\lambda\theta) + B\cos(\lambda\theta).$$

To get exactly a period of 2π, we must take $\lambda = n$ to be a positive integer.

Now our second equation takes the form

$$T'(t) = -n^2\alpha^2 T(t).$$

If $n > 0$, this has solution $T(t) = e^{-n^2\alpha^2 t}$. If $n = 0$, this has solution $T(t) = A_0$, a constant.

So, our solution to this problem is given by

$$u(t,\theta) = A_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right),$$

where the coefficients are chosen to meet the final boundary condition

$$f(\theta) = u(0,\theta) = A_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right).$$

That is, the coefficients are the Fourier coefficients of f.

Now, we see that in the limit as $t \to \infty$, all of the terms having exponentials die off. The limiting value is then $u(\infty,\theta) = A_0$, a constant. It means that the final temperature distribution is that every point has the same equilibrium temperature of A_0. This makes good physical sense because A_0 is the average temperature of the initial distribution $f(\theta)$.

Problem 5: First we handle part (a). It will make things faster if we agree to write $\xi = x^2 + y^2 + z^2$. We compute that

$$\frac{\partial u}{\partial x} = -x\xi^{-3/2}, \quad \frac{\partial u}{\partial y} = -y\xi^{-3/2}, \quad \frac{\partial u}{\partial z} = -z\xi^{-3/2},$$

and that

$$\frac{\partial^2 u}{\partial x^2} = -\xi^{-3/2} + 3x^2\xi^{-5/2}, \quad \frac{\partial^2 u}{\partial y^2} = -\xi^{-3/2} + 3y^2\xi^{-5/2}, \quad \frac{\partial^2 u}{\partial z^2} = -\xi^{-3/2} + 3z^2\xi^{-5/2}.$$

So that

$$\triangle u = -3\xi^{-3/2} + 3\xi^{-5/2}(x^2 + y^2 + z^2) = 0.$$

Now consider part (b). Using the expression for the Laplacian in spherical coordinates, we see that

$$\triangle u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}.$$

But for us,

$$\frac{\partial u}{\partial\theta} = 0 = \frac{\partial^2 u}{\partial\varphi^2}.$$

And it is easy to see that

$$\frac{\partial u}{\partial r} = \frac{-1}{r^2}.$$

So, putting in this information, we get

$$\begin{aligned}\triangle u &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + 0\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{-1}{r^2}\right)\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}(-1) = 0.\end{aligned}$$

Finally, consider part (c). The key here is that taking partial derivatives of v is essentially the same as taking them of u. We use the chain rule a couple of times and the relevant bit is that

$$\frac{d(x-x_0)}{dx} = 1, \quad \frac{d(x-x_0)}{dy} = 0, \quad \frac{d(x-x_0)}{dz} = 0.$$

And similar things happen for $y - y_0$ and $z - z_0$. So in the long run, $\triangle v = \triangle u = 0$.

Problem 6: If you use the recurrence formula, you get that

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}\left(3x^2 - 1\right) \\
P_3(x) &= \frac{1}{2}\left(5x^3 - 3x\right) \\
P_4(x) &= \frac{1}{8}\left(35x^4 - 30x^2 + 3\right) \\
P_5(x) &= \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right) \\
P_6(x) &= \frac{1}{48}\left(693x^6 - 945x^4 + 315x^2 - 15\right) \\
P_7(x) &= \frac{1}{48}\left(1287x^7 - 2079x^5 + 945x^3 - 105x\right).
\end{aligned}$$

Graphs are now easy to produce. Notice that even though the coefficients look pretty large, the functions are still not too big on the interval $(-1, 1)$.