**Solutions to Homework # 5**
**Math 381, Rice University, Fall 2003**

**Problems 1-4:** Since these are polynomials, it is not to hard to compute the answers. First, note that a Legendre polynomial is orthogonal to any polynomial of lower degree, so most of the coefficients will disappear. Then you can explicitly calculate the few that are left by either using the definition and integrating, or by using a little linear algebra like we did in class. The answers are

$$\begin{aligned}
f(x) = x^2 &= \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x), \\
g(x) = x^3 &= \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x), \\
h(x) = x^4 &= \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) - \frac{1}{5}P_0(x), \\
j(x) = x^5 &= \frac{8}{63}P_5(x) + \frac{4}{9}P_3(x) + \frac{3}{7}P_1(x),
\end{aligned}$$

**Problem 5:** First note that the function is even, so for $k$ odd, we know $a_k = 0$. We only need to find the even coefficients. To compute this example, it is easiest to use the recurrence formula.

$$(2k+1)xP_k(x) = kP_{k-1}(x) + (k+1)P_{k+1}(x).$$

Using this and the fact that our function is even, we find the $k$-th Legendre coefficient to be

$$\begin{aligned}
a_k &= \frac{2k+1}{2}\int_{-1}^{1} |x|P_k(x)\ dx \\
&= \int_0^1 (2k+1)xP_k(x)\ dx \\
&= \int_0^1 \left(kP_{k-1}(x) + (k+1)P_{k+1}(x)\right)\ dx \\
&= k\int_0^1 P_{k-1}(x)\ dx + (k+1)\int_0^1 P_{k+1}(x)\ dx.
\end{aligned}$$

But we figured out how to compute these funny integrals in class. (Also, they can be found in your Textbook on page 234.) They are given by

$$\int_0^1 P_n(x)\ dx = \begin{cases} 1 & n = 0 \\ 0 & n = 2, 4, 6, \ldots \\ (-1)^{(n-1)/2}\dfrac{1}{n(n+1)}\dfrac{(1\cdot 3\cdot 5\cdots n)^2}{n!} & n = 1, 3, 5, \ldots \end{cases}$$

Recall that this was obtained while computing an in-class example. Since we are only interested in $k$ even, we only get integrals where the index is odd.

Substituting this into our expression above, we see

$$\begin{aligned}
a_k &= k\int_0^1 P_{k-1}(x)\ dx + (k+1)\int_0^1 P_{k+1}(x)\ dx \\
&= (-1)^{(k-2)/2}\frac{k}{(k-1)k}\frac{(1\cdot 3\cdot 5\cdots(k-1))^2}{(k-1)!} + (-1)^{k/2}\frac{(k+1)}{(k+1)(k+2)}\frac{(1\cdot 3\cdot 5\cdots(k+1))^2}{(k+1)!} \\
&= (-1)^{(k-2)/2}\frac{(1\cdot 3\cdot 5\cdots(k-1))^2}{(k-1)!}\left[\frac{1}{k-1} - \frac{(k+1)^2}{k(k+1)(k+2)}\right] \\
&= (-1)^{(k-2)/2}\frac{(1\cdot 3\cdot 5\cdots(k-1))^2}{(k-1)!}\left[\frac{1}{k-1} - \frac{(k+1)}{k(k+2)}\right] \\
&= (-1)^{(k-2)/2}\frac{(1\cdot 3\cdot 5\cdots(k-1))^2}{(k-1)!}\left[\frac{2k+1}{(k-1)k(k+2)}\right] \\
&\text{or} \ldots \\
&= (-1)^{k/2+1}(2k+1)\frac{1\cdot 3\cdot 5\cdots(k-3)}{2\cdot 4\cdot 6\cdots k(k+2)}.
\end{aligned}$$

This final expression doesn't make sense for $k = 0, 2$, but we can find those from the recurrence formula by hand pretty quickly. For doing the rest of the problem, we get that the 6th order partial sum is

$$s_6(x) = \frac{1}{2}P_0(x) + \frac{5}{8}P_2(x) - \frac{3}{16}P_4(x) + \frac{13}{128}P_6(x).$$

This allows us to make nice graphs.

**Problems 6-7:** First we know that the Taylor polynomial of degree 10 for $f(x) = \cos(2\pi x)$ is

$$Q_{10}(x) = 1 - \frac{4\pi^2}{2}x^2 + \frac{16\pi^4}{24}x^4 - \frac{64\pi^6}{720}x^6 + \frac{256\pi^8}{40320}x^8 - \frac{1024\pi^{10}}{362880}x^{10}.$$

Next, we compute the order 10 partial sum of the Legendre series for $f(x)$. I used *Matlab* to find the following:

$$\begin{aligned}
\int_{-1}^1 x^2\cos(2\pi x)\ dx &= \frac{1}{\pi^2} \\
\int_{-1}^1 x^4\cos(2\pi x)\ dx &= \frac{2}{\pi^2} - \frac{3}{\pi^4} \\
\int_{-1}^1 x^6\cos(2\pi x)\ dx &= \frac{3}{\pi^2} - \frac{15}{\pi^4} + \frac{45}{2\pi^6} \\
\int_{-1}^1 x^8\cos(2\pi x)\ dx &= \frac{4}{\pi^2} - \frac{42}{\pi^4} + \frac{210}{\pi^6} - \frac{315}{\pi^8} \\
\int_{-1}^1 x^{10}\cos(2\pi x)\ dx &= \frac{5}{\pi^2} - \frac{90}{\pi^4} + \frac{945}{\pi^6} - \frac{4725}{\pi^8} + \frac{14175}{2\pi^{10}}
\end{aligned}$$

With these in hand, we can compute the Legendre coefficients of $f(x)$ as

$$\begin{aligned}
a_2 =& \frac{15}{4\pi^2} \approx 0.38 \\
a_4 =& \frac{45}{2\pi^2} - \frac{945}{16\pi^4} \approx 1.0671 \\
a_6 =& \frac{2184}{32\pi^2} - \frac{32760}{32\pi^4} + \frac{135135}{64\pi^6} \approx -1.3983 \\
a_8 =& \frac{153}{\pi^2} - \frac{58905}{8\pi^4} + \frac{2297295}{32\pi^6} - \frac{34459425}{256\pi^8} \approx 0.3998 \\
a_{10} \approx& -.0601
\end{aligned}$$

Again, I used *Matlab* as a calculator. Here, it is helpful to notice that $\cos(2\pi)$ is even and has average zero, so $a_0 = 0$ and $a_k = 0$ for $k$ odd. Thus our 10th order partial sum is

$$s_{10}(x) = a_2 P_2(x) + a_4 P_4(x) + a_6 P_6(x) + a_8 P_8(x) + a_{10} P_{10}(x).$$

This allows us to make the graphs required for problem 6. Note that $s_1 0$ is hard to distinguish from $f(x)$ with the naked eye.

For problem 7, note that the Taylor polynomial and $s_{10}$ are both polynomials of degree 10, though they have complicated coefficients. To compare them it is best to look at their graphs. Again, I used Matlab. What becomes clear is that the Taylor polynomials are very good near $x = 0$, but bad everywhere else. The Legendre series is pretty close just about everywhere on the interval $(-1, 1)$. This matches our expectations, because this is how these things are defined.

At this point, two things should be clear to you. First, like Fourier series, Legendre series are good for approximations over the interval where they are orthogonal. Second, Legendre series of even a simple function are hard to compute by hand, so it is best to use a computer.

I'll attach the graphs for problems 1-7 at the end of this document.

**Bonus Problem:** The proof here runs exactly like the case for Fourier series.

Fix our function $f$ such that $f$ and $f^2$ are integrable on $(-1, 1)$. Let $s_n(x)$ denote the $n$th order partial sum of the Legendre series for $f$ at $x$. Also, let $p(x)$ be any polynomial of degree at most $n$.

Let $R_n(x) = f(x) - s_n(x)$ be the error. Note that $s_n(x)$ is its own Legendre series because it is a polynomial. We have also that

$$\int_{-1}^{1} R_n(x) q(x)\, dx = 0$$

for all polynomials $q(x)$ of degree at most $n$. You can see this by direct computation, or by writing $R_n$ as a Legendre series and using orthogonality.

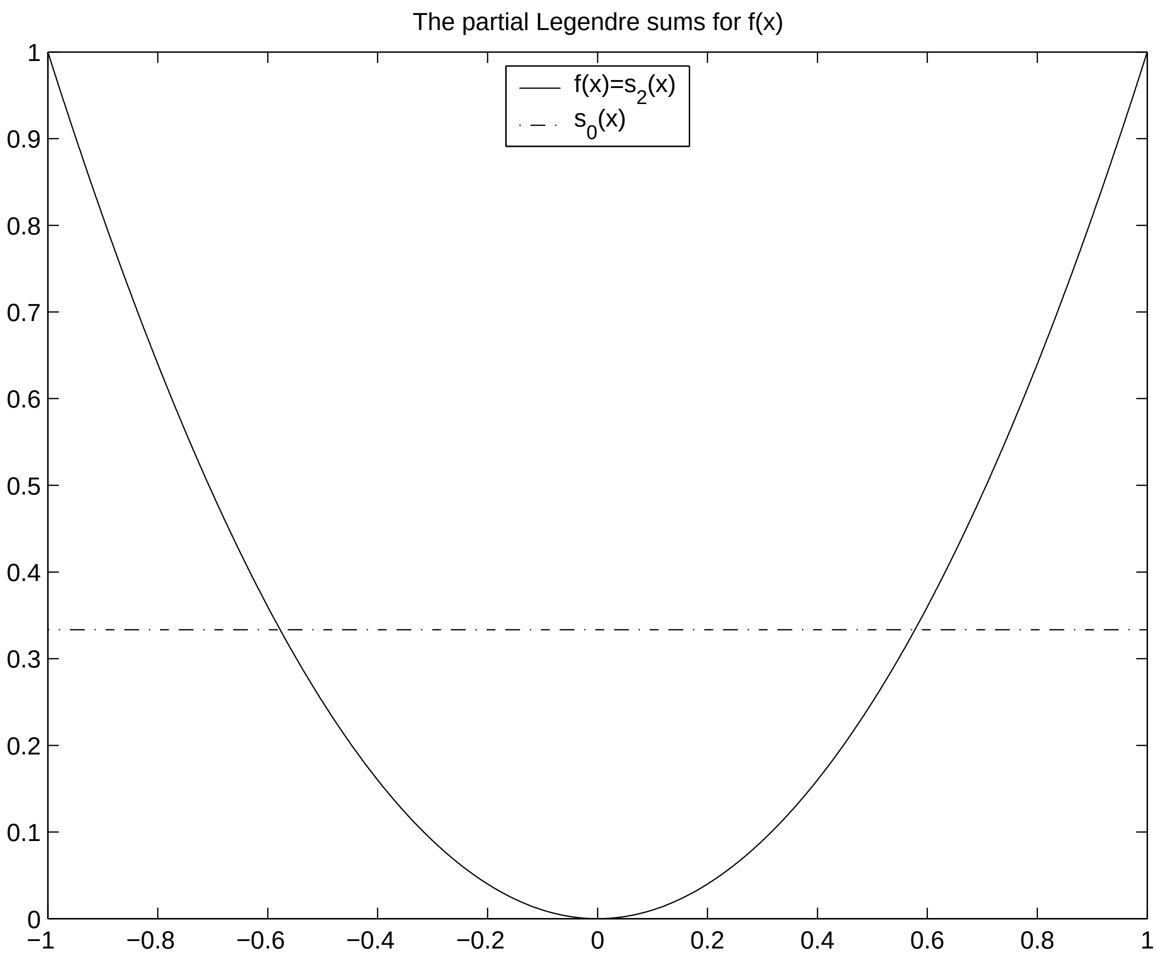
Now we have that

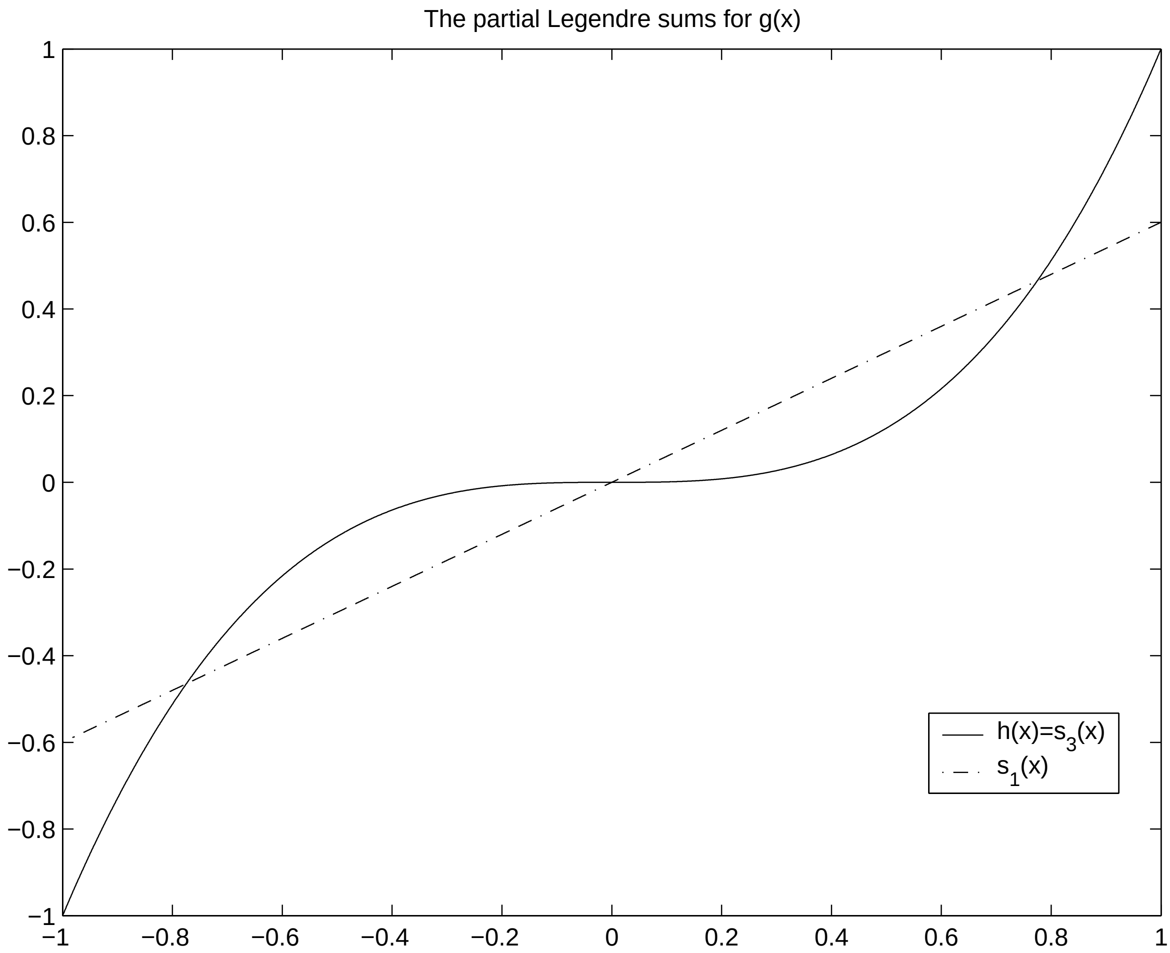
$$\begin{aligned}\int_{-1}^{1}(f(x)-p(x))^2\ dx &= \int_{-1}^{1}(R_n(x)+s_n(x)+p(x))^2\ dx\\ &= \int_{-1}^{1} R_n(x)^2\ dx - 2\int_{-1}^{1} R_n(x)(p(x)-s_n(x))\ dx + \int_{-1}^{1}(p(x)-s_n(x))^2\ dx\end{aligned}$$

The second term in this last expression is zero by the remark above because $p(x)-s_n(x)$ is a polynomial of degree at most $n$. The third term is the integral of a non-negative function, so must be non-negative. This means that

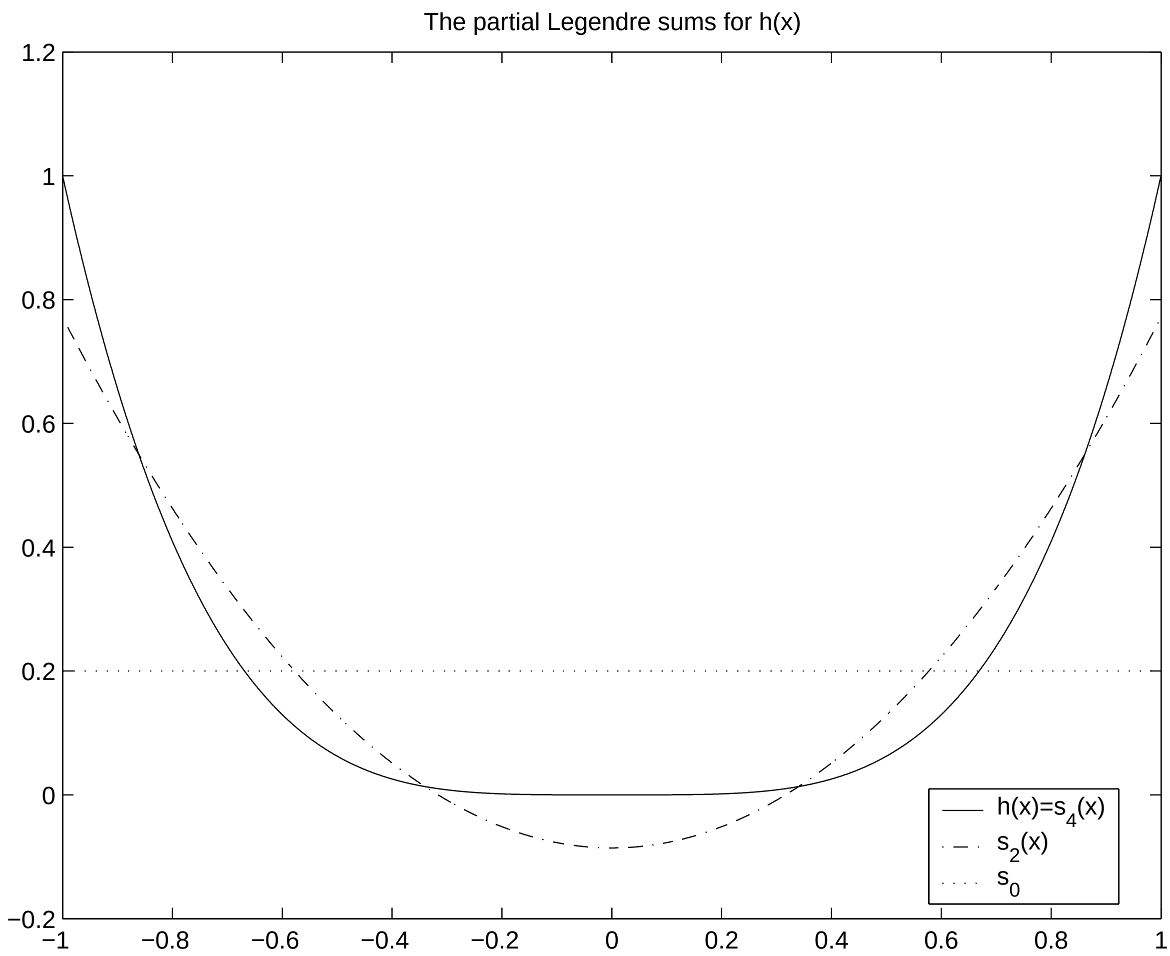
$$\int_{-1}^{1}(f(x)-p(x))^2\ dx \geq \int_{-1}^{1} R_n(x)^2\ dx = \int_{-1}^{1}(f(x)-s_n(x))^2\ dx.$$

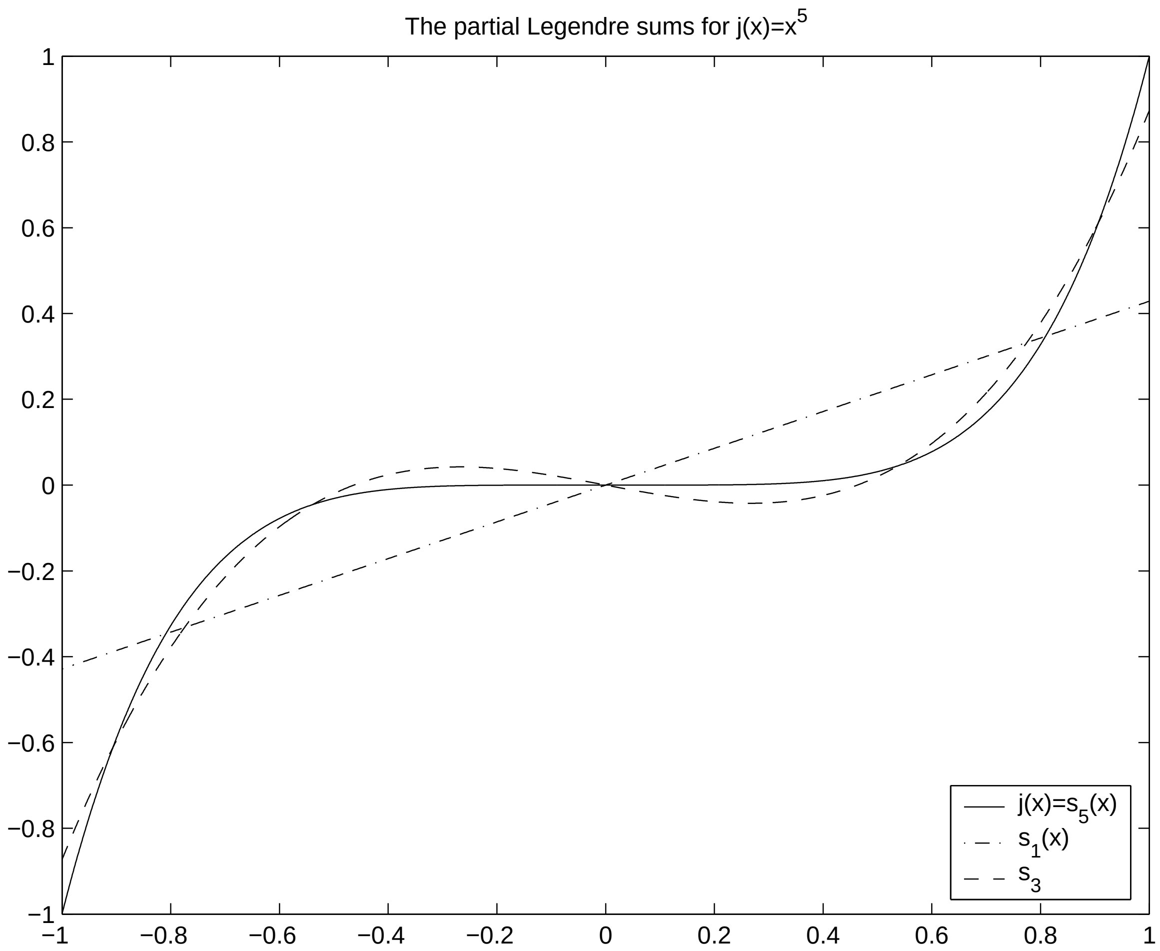
This completes the proof. Notice that equality holds only if that third term vanishes, that is if $p(x)\equiv s_n(x)$.

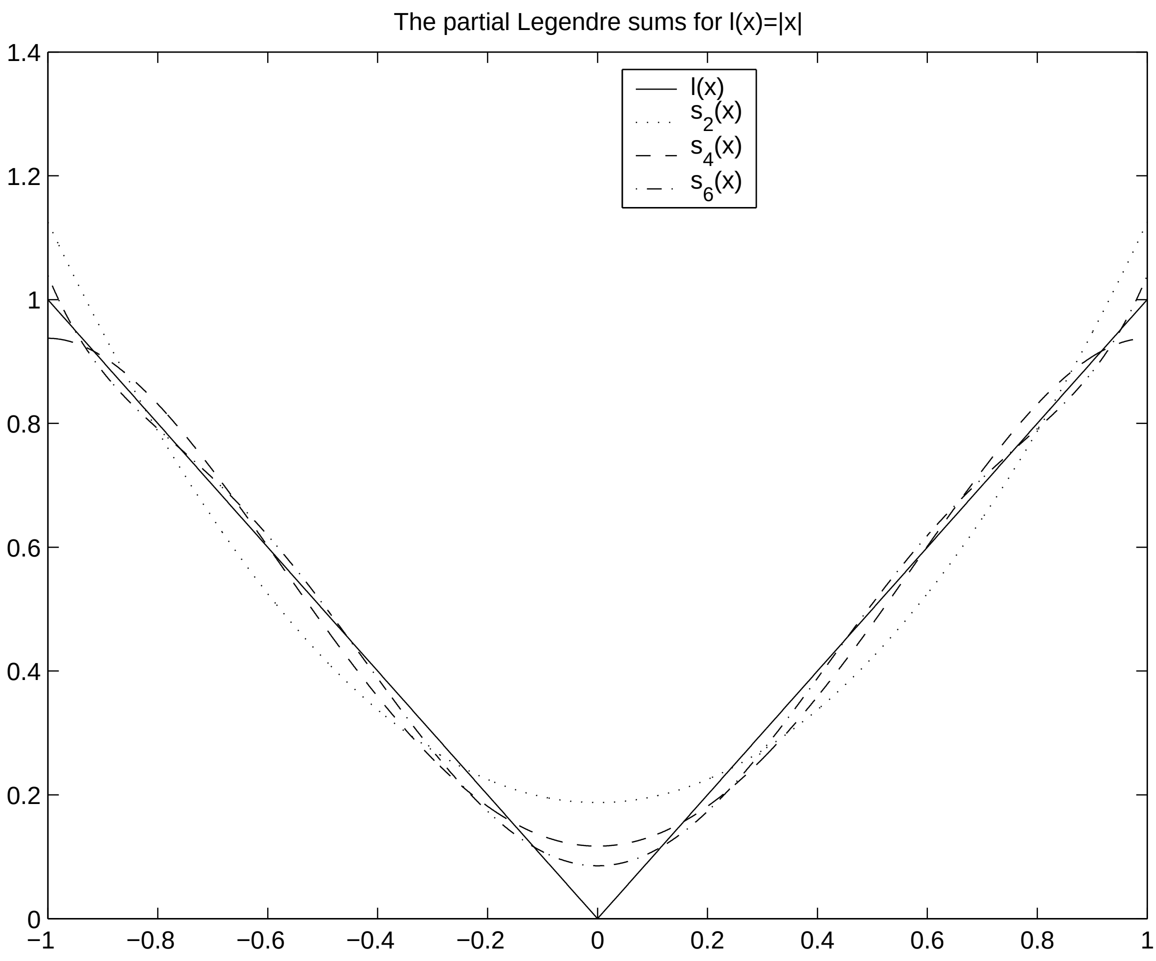

The partial Legendre sums for f(x)
f(x)=s2(x)
s0(x)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

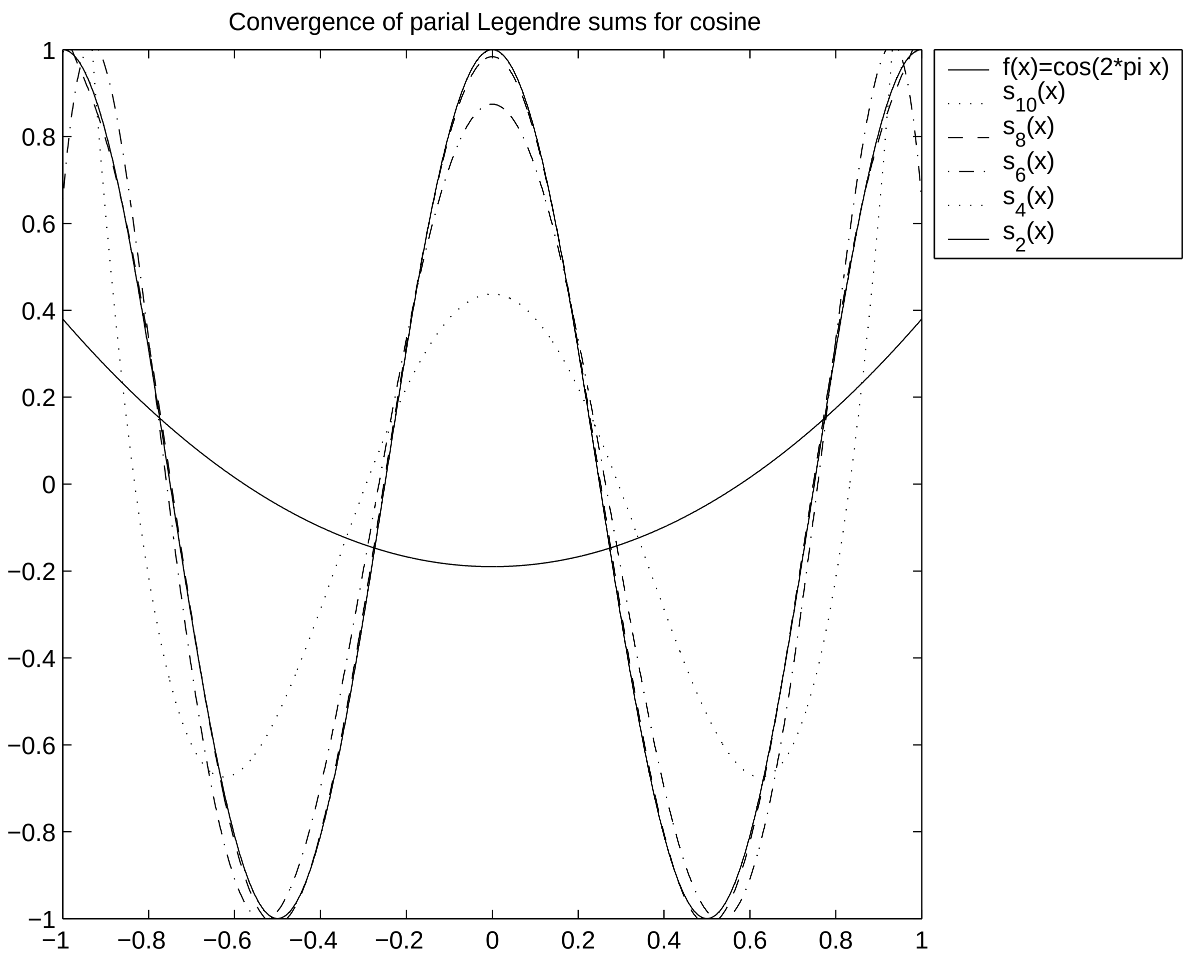


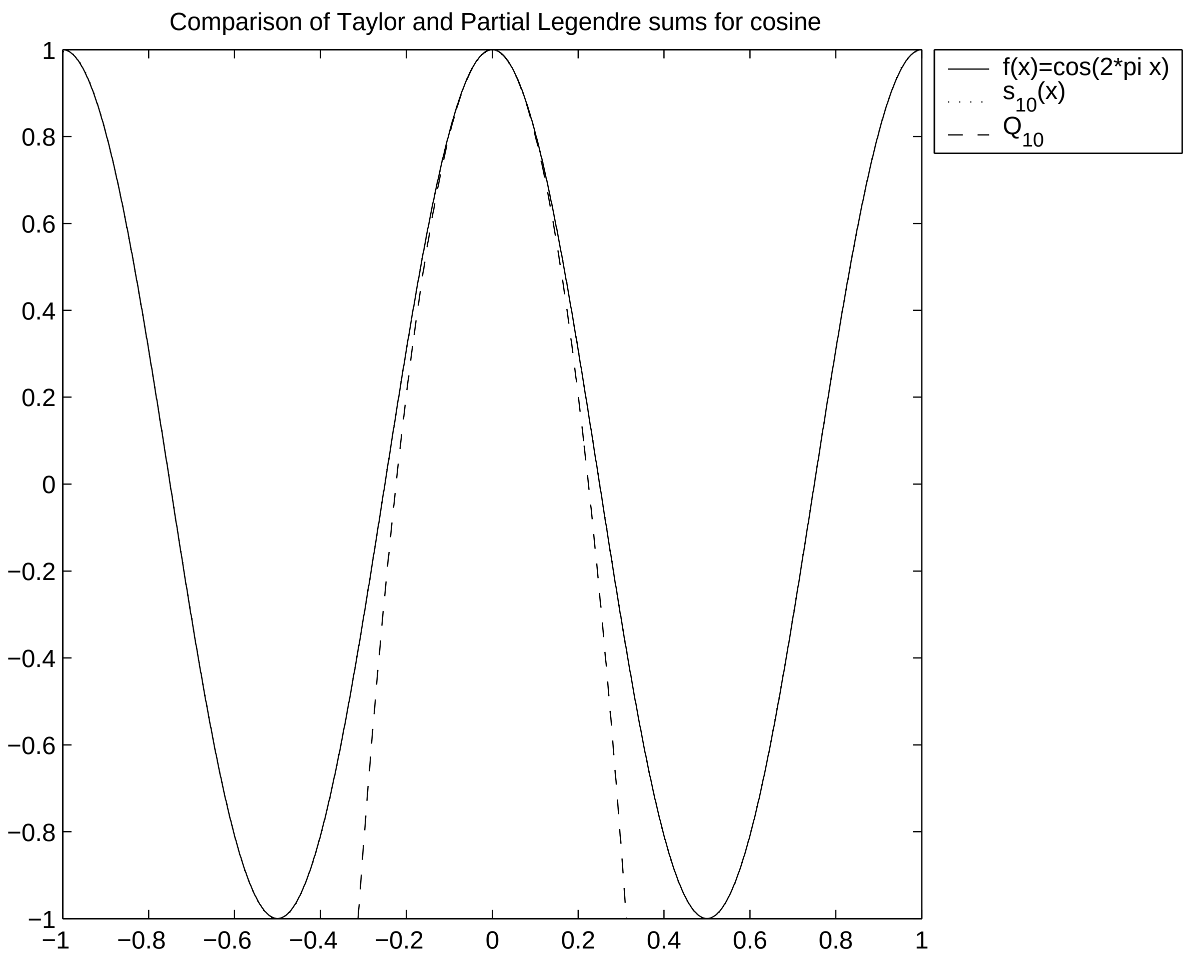
The partial Legendre sums for g(x)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
h(x)=s3(x)
s1(x)


The partial Legendre sums for h(x)
h(x)=s4(x)
s2(x)
s0
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


The partial Legendre sums for j(x)=x^5
j(x)=s_5(x)
s_1(x)
s_3
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


The partial Legendre sums for l(x)=|x|
l(x)
s2(x)
s4(x)
s6(x)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


Convergence of parial Legendre sums for cosine
f(x)=cos(2*pi x)
s10(x)
s8(x)
s6(x)
s4(x)
s2(x)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


Comparison of Taylor and Partial Legendre sums for cosine
f(x)=cos(2*pi x)
$s_{10}(x)$
$Q_{10}$
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1