Homework Assignment # 6
Due Friday, November 21, 2003.

Problem 1: Find a function which is harmonic inside the sphere of radius 1 and which on the boundary sphere is equal to

$$g(x, y, z) = \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)^3 .$$

Problem 2: Perform separation of variables for the general ("unsymmetric") Laplace equation in spherical coordinates. You need only give the set of ordinary differential equations as your result.

Problem 3: Show that if $\lambda_{n,1} < \lambda_{n,2} < \ldots$ are the zeros of the Bessel function of order n, then the functions

$$J_n(\lambda_{n,p} x) \cos(ny), \quad J_m(\lambda_{m,q} x) \sin(my), \quad n, m, p, q \text{ are non-negative integers}$$

form an orthogonal family over the region $0 \leq x \leq 1$, $-\pi \leq y \leq \pi$ with respect to the weight function $\rho(x, y) = x$.

Problem 4: Let J_0 be the Bessel function of order zero and the first kind. Evaluate the integral $\int_0^\lambda x J_0(x)\, dx$ for an arbitrary $\lambda > 0$.

Problem 5: Expand the function $f(x) = 1$ in a Bessel series of order zero on the roots of the equation $J_0(x) = 0$. Your answer should be written in terms of the roots $\lambda_1 < \lambda_2 < \ldots$ of J_0.

Problem 6: Let $\mathcal{C}$ be the region bounded by the cylinder of radius one with the z-axis as its "center" curve and the planes $z = 0$ and $z = A$. Find a function which is harmonic inside $\mathcal{C}$, takes the value 0 on the sides and top of the cylinder, and the value $f(x, y, 0) = 1$ on the bottom.

Problem 7: What should the analog of Parseval's theorem for Bessel series of order zero be? Be careful about the weight function!

Problem 8: Use power series to prove the following recurrence relations for the Bessel functions $J_n(x)$.

- $J_n'(x) = \dfrac{1}{2} \left(J_{n-1}(x) - J_{n+1}(x) \right)$
- $J_{n+1}(x) + J_{n-1}(x) = \dfrac{2n}{x} J_n(x)$

Problem 9: Consider the following list of second order ordinary differential equations. Determine which are of "Sturm-Liouville" type. For those which are of Sturm-Liouville type, rewrite the equation to fit the standard form

$$-[p(x)y']' + q(x)y = \lambda r(x)y$$

determine the appropriate weight function for orthogonality of solutions. Which of these would define regular Sturm-Liouville problems, and which would define singular problems?

Chebyshev's equation $(1-x^2)y'' - xy' + \alpha^2 y = 0$ for $-1 < x < 1$, where α is a constant.

Airy's equation $y'' = k^2 xy$ for $-\infty < x < \infty$, where k is a constant.

Hermite's equation $y'' = 2xy' - \lambda y$ for $-\infty < x < \infty$, where λ is a constant.

Laguerre's equation $xy'' + (1-x)y' + \lambda y = 0$ for $0 < x < \infty$.

Gauss' hypergeometric equation $x(1-x)y'' + [\gamma - (1+\alpha+\beta)x]y' - \alpha\beta y = 0$ for $0 < x < \infty$ where α, β, γ are constants.

Super Bonus Problem: Is there a 'Gibbs phenomenon' for the Bessel series of order zero or Legendre series? It is OK to pick a nice function and work only with it. For example, consider what happens near zero for the Legendre series of

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 \le x < 1. \end{cases}$$