Solutions to Homework # 6
Math 381, Rice University, Fall 2003

Problem 1: The problem is about Laplace's equation in a sphere, so we shall use spherical coordinates (r, φ, θ). First, we translate our boundary condition. Note that in spherical coordinates

$$g(r, \varphi, \theta) = g(x, y, z) = \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)^3 = \left(\frac{r \cos \theta}{r} \right)^3 = \cos^3(\theta).$$

The important thing to notice is that this is independent of the longitudinal coordinate φ. This means that we can use our solution from class!

Applying our in-class work, we know that the solution to this problem is given as

$$u(r, \theta, \varphi) = u(r, \theta) = \sum_{n=0}^{\infty} \alpha_n P_n(\cos \theta) r^n$$

where the α_n's are chosen to be the Legendre series coefficients of the function $f(x) = g(1, \cos^{-1} \theta) = x^3$. In homework assignment # 4 we found that the Legendre series for x^3 is

$$x^3 = \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x).$$

Putting all this together, we see that our solution is

$$\begin{aligned} u(r, \theta) &= \frac{2}{5} P_3(\cos \theta) r^3 + \frac{3}{5} P_1(\cos \theta) r \\ &= \frac{2r^3}{5} \left(\frac{5}{2} \cos^3 \theta - \frac{3}{2} \cos \theta \right) + \frac{3r}{5} (\cos \theta) \\ &= (r \cos \theta)^3 - \frac{3}{5} (r^2 - 1) r \cos \theta. \end{aligned}$$

Since the original problem was stated in rectangular coordinates, we change back to find our solution is

$$u(x, y, z) = z^3 - \frac{3z(x^2 + y^2 + z^2 - 1)}{5}.$$

It is not difficult to check that this agrees with g on the boundary of the sphere, especially because this boundary is the surface $r = \sqrt{x^2 + y^2 + z^2} = 1$.

Problem 2: Recall from homework 4 that in spherical coordinates (r, θ, φ), the Laplace equation takes the form

$$\triangle u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}.$$

In order to perform the separation of variables technique, we assume that u has the form of a product

$$u(r, \theta, \varphi) = R(r) \cdot T(\theta) \cdot F(\varphi).$$

Substituting this into the equation $0 = \triangle u$, we see that the functions R, T, F must satisfy the following rule:

$$\begin{aligned} 0 &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial RTF}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial RTF}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 RTF}{\partial\varphi^2} \\ &= \frac{TF}{r^2}\frac{\partial}{\partial r}\left(r^2R'\right) + \frac{RF}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta T'\right) + \frac{RTF''}{r^2\sin^2\theta} \\ &= \frac{TF}{r^2}\left(r^2R'' + 2rR'\right) + \frac{RF}{r^2\sin\theta}\left(\sin\theta T'' + \cos\theta T'\right) + \frac{RTF''}{r^2\sin^2\theta} \end{aligned}$$

First, we separate out the φ variable. We rearrange the last equation to the equivalent expression

$$-\frac{F''}{F} = \frac{r^2\sin^2\theta}{RT}\left(\frac{T}{r^2}\left(r^2R'' + 2rR'\right) + \frac{R}{r^2\sin\theta}\left(\sin\theta T'' + \cos\theta T'\right)\right)$$

The left hand side of this equation is independent of r and θ and the right hand side is independent of φ; therefore, this expression must be constant. Denoting this constant by C, we get two equations.

$$\begin{aligned} F'' &= -CF, \\ C &= \frac{r^2\sin^2\theta}{RT}\left(\frac{T}{r^2}\left(r^2R'' + 2rR'\right) + \frac{R}{r^2\sin\theta}\left(\sin\theta T'' + \cos\theta T'\right)\right) \end{aligned}$$

To make things a bit easier to read, we now simplify the second equation. The simplified version is

$$C = \frac{\sin^2\theta}{R}\left(r^2R'' + 2rR'\right) + \frac{\sin\theta}{T}\left(\sin\theta T'' + \cos\theta T'\right).$$

Next, we separate the two variables which remain here.

$$\frac{r^2R'' + 2rR'}{R} = -\frac{T'' + \cot\theta T' - C\csc^2\theta T}{T}.$$

Now the left hand side is independent of θ and the right hand side is independent of r. Again, this quantity must be a constant. We denote this constant by A.

Putting all of this information together, we have the following three ordinary differential equations to solve.

$$\begin{aligned} F'' + C \cdot F &= 0 \\ r^2 R'' + 2rR' - A \cdot R &= 0 \\ T'' + \cot\theta T' \left(A - \frac{C}{\sin^2\theta} \right) T &= 0. \end{aligned}$$

Notice that if our solution is assumed to be independent of φ, then we get that $C = 0$ and the situation reduces to the one we studied in class.

Problem 3: Every single integral that one needs to check here is of the form

$$\int_{-\pi}^{\pi} \int_{0}^{1} A(x)B(y)\ dx\ dy,$$

where $A(x)$ is a product of Bessel functions and $B(y)$ is a product of trig functions. If we do the integration with respect to y first, we see by the orthogonality of sines and cosines that the answer will be zero unless $m = n$ and the two functions are either both sin or both cos. In the case where $m = n$ and the functions agree, we get a positive constant (actually π) times the integral

$$\int_{0}^{1} J_n(\lambda_{n,p}x) \cdot J_n(\lambda_{n,q}x)\ dx.$$

But by the orthogonality properties of Bessel functions, this is zero unless $p = q$. Therefore, if we take any two functions from the family described in the problem which are different, we get that they are orthogonal.

Problem 4: Since J_0 is a solution of Bessel's equation of order zero, we see that

$$J_0''(x) + \frac{1}{x}J_0'(x) + J_0(x) = 0.$$

Multiplying through by x we find

$$0 = x \cdot J_0''(x) + J_0'(x) + x \cdot J_0(x) = [x \cdot J_0'(x)]' + x \cdot J_0(x).$$

Therefore, by the Fundamental Theorem of Calculus,

$$\int_{0}^{\lambda} xJ_0(x)\ dx = -\lambda \cdot J_0'(\lambda).$$

Problem 5: Using the last problem, we see that the kth coefficient is

$$\begin{aligned}\alpha_k =& \frac{2}{J_0'(\lambda_k)^2}\int_0^1 xJ_0(\lambda_k x)\ dx\\ =& \frac{2}{\lambda_k^2 J_0'(\lambda_k)^2}\int_0^{\lambda_k} uJ_0(u)\ du\\ =& \frac{2}{\lambda_k^2 J_0'(\lambda_k)^2}\left(-\lambda_k J_0'(\lambda_k)\right)\\ =& \frac{-2}{\lambda_k J_0'(\lambda_k)}.\end{aligned}$$

Therefore the series is

$$1 = f(x) = \sum_{k=1}^{\infty} \frac{-2}{\lambda_k J_0'(\lambda_k)} J_0(\lambda_k x)$$

Problem 6: Note that the boundary conditions on this problem are rotationally symmetric! Therefore, by the separation of variables work we did in class, we know that a product solution of the form will be $u(r,z) = Z(z)\cdot R(r)$, where Z and R solve the equations

$$Z'' = -C\cdot Z, \qquad R'' + \frac{1}{r}R' - C\cdot R = 0.$$

In order to meet the boundary conditions at $z = A$ and $z = 0$, it is most appropriate to use exponentials to solve the equation for Z. Thus we take $C = -\lambda^2 < 0$. Then we get that

$$Z(z) = Be^{\lambda z} + De^{-\lambda x}, \qquad R(r) = J_0(\lambda r).$$

In order that the condition of vanishing on the sides of the cylinder, we should choose λ to be a zero of J_0. To make the function vanish at the top of the cylinder, we need to choose B and D so that $Z(z) = \sinh(\lambda(A-z))$.

So, if we let $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ be the roots of $J_0(x) = 0$, then our solution looks like

$$u(r,z) = \sum_{i=1}^{\infty} c_i \sinh(\lambda_i(A-z))J_0(\lambda_i r).$$

To meet our final boundary condition, we want

$$1 = f(r) = \sum_{i=1}^{\infty} c_i \sinh(\lambda_i A)J_0(\lambda_i r).$$

This is just a Bessel series expansion! So we only need to choose c_i to be the ith Bessel series coefficient divided by $\sinh(\lambda_i A)$. We computed these coefficients in the last problem, so we get a final solution of

$$u(r,z) = \sum_{i=1}^{\infty} \frac{-2}{\lambda_i J_0'(\lambda_i)\sinh(\lambda_i A)} \sinh(\lambda_i(A-z))J_0(\lambda_i r)$$

Problem 7: For Bessel series of order zero indexed on the roots of J_0, the analog of Parseval's theorem would read as follows:

Theorem 1 *Let f be a continuous function on the interval $[0,1]$. Order the roots of the equation $J_0(x) = 0$ as $\lambda_1 < \lambda_2 < \lambda_3 < \dots$. Let*

$$\alpha_k = \frac{2}{J_0(\lambda_k)^2 + J_0'(\lambda_k)^2} \int_0^1 rf(r)J_0(\lambda_k r)\ dr$$

be the kth coefficient of the Bessel series of order zero of f. Then

$$\sum_{k=1}^{\infty} \frac{J_0(\lambda_k)^2 + J_0'(\lambda_k)^2}{2} \alpha_k^2 = \int_0^1 x[f(x)]^2\ dx.$$

Again, this should be interpreted as showing that the weighted mean square error goes to zero, because we can calculate the weighted mean square error of the nth partial sum as the difference between the term on the right and the nth partial sum of the series on the left.

It is also possible to go with the hypothesis "f and f^2 are integrable on $(0,1)$" instead of "f is continuous".

Problem 8: It is helpful in this problem to have a closed form for the infinite sum which represents $J_n(x)$. This is

$$\begin{aligned} J_n(x) &= \frac{x^n}{2^n \cdot n!} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} \cdot k! \cdot (n+1) \cdots (n+k)} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{n+2k}}{2^{n+2k} \cdot k! \cdot (n+k)!}. \end{aligned}$$

Using this, we can then compute that

$$\begin{aligned} &\frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right) = \\ &= \frac{1}{2}\left(\sum_{l=0}^{\infty} \frac{(-1)^l x^{n+2l-1}}{2^{n+2l-1} \cdot l! \cdot (n+l-1)!} - \sum_{l=1}^{\infty} \frac{(-1)(-1)^l x^{n+2l-1}}{2^{n+2l-1} \cdot (l-1)! \cdot (n+l)!} \right) \\ &= \frac{1}{2}\left(\frac{x^{n-1}}{2^{n-1} \cdot (n-1)!} + \sum_{l=1}^{\infty} \frac{(-1)^l x^{n+2l-1}}{2^{n+2l-1}} \left[\frac{1}{l! \cdot (n+l-1)!} + \frac{1}{(l-1)! \cdot (n+l)!} \right] \right) \\ &= \frac{nx^{n-1}}{2^n \cdot n!} + \sum_{l=1}^{\infty} \frac{(-1)^l x^{n+2l-1}}{2^{n+2l}} \frac{(n+l)+l}{l! \cdot (n+l)!} \\ &= \sum_{l=0}^{\infty} \frac{(n+2l)(-1)^k x^{n+2l-1}}{2^{n+2l} \cdot l! \cdot (n+l)!} = J_n'(x) \end{aligned}$$

Note that to get the first line, we re-indexed the sum for J_{n+1} so that the terms would line up with those of J_{n-1}. This proves the first of the two desired relations.

For the second equality, we proceed in a similar fashion.

$$\begin{aligned}
J_{n-1}(x) + J_{n+1}(x) &= \sum_{l=0}^{\infty} \frac{(-1)^l x^{n+2l-1}}{2^{n+2l-1} \cdot l! \cdot (n+l-1)!} + \sum_{l=1}^{\infty} \frac{(-1)(-1)^l x^{n+2l-1}}{2^{n+2l-1} \cdot (l-1)! \cdot (n+l)!} \\
&= \frac{x^{n-1}}{2^{n-1} \cdot (n-1)!} + \sum_{l=1}^{\infty} \frac{(-1)^l x^{n+2l-1}}{2^{n+2l-1}} \left[\frac{1}{l! \cdot (n+l-1)!} - \frac{1}{(l-1)! \cdot (n+l)!} \right] \\
&= \frac{n x^{n-1}}{2^{n-1} \cdot n!} + \sum_{l=1}^{\infty} \frac{(-1)^l x^{n+2l-1}}{2^{n+2l-1}} \frac{(n+l) - l}{l! \cdot (n+l)!} \\
&= \sum_{l=0}^{\infty} \frac{n(-1)^k x^{n+2l-1}}{2^{n+2l-1} \cdot l! \cdot (n+l)!} \\
&= \frac{2n}{x} \sum_{l=0}^{\infty} \frac{(-1)^l x^{n+2l}}{2^{n+2l} \cdot l! \cdot (n+l)!} = \frac{2n}{x} J_n(x)
\end{aligned}$$

This proves the second formula, and we are done.

Problem 9: All of these equations give rise to Sturm-Liouville problems.

To handle Chebyshev's equation, multiply through by $(1-x^2)^{-1/2}$. Then you find that $p(x) = \sqrt{1-x^2}$, $q(x) = 0$ and the weight function is $r(x) = (1-x^2)^{-1/2}$. This problem is singular because p and r are 'bad' at the boundary. In fact, r isn't even continuous on the closed interval $[-1, 1]$ because it blows up.

$$-[\sqrt{1-x^2} \cdot y']' = \lambda(1-x^2)^{-1/2} \cdot y$$

For Airy's equation, we see that $p(x) = 1$, $q(x) = 0$ and our weight function is $r(x) = x$. Here, r blows up at the boundary points $\pm\infty$, so the problem is singular.

$$-[y']' = \lambda x \cdot y$$

Hermite's equation requires multiplying through by e^{-x^2} to put it into standard form. We get that $p(x) = e^{-x^2}$, $q(x) = 0$ and the weight function is $r(x) = e^{-x^2}$. Both p and r vanish at the boundary points $\pm\infty$, so this problem is singular.

$$-[e^{-x^2} \cdot y']' = \lambda e^{-x^2} \cdot y$$

For Laguerre's equation, one must multiply through by e^{-x}. Then we find $p(x) = xe^{-x}$, $q(x) = 0$ and the weight function is $r(x) = e^{-x}$. Again, this problem is singular because p and r vanish at endpoint $+\infty$ (in fact, p also vanishes at $x = 0$).

$$-[xe^{-x} \cdot y']' = \lambda e^{-x} \cdot y$$

Finally, for Gauss' hypergeometric equation, we must multiply through by $x^{\gamma-1}(1-x)^{\alpha+\beta-\gamma}$ (I did the integral that results by using partial fractions). Then $p(x) = x^{\gamma}(1-x)^{1+\alpha+\beta-\gamma}$, $q(x) = 0$ and the weight function is $r(x) = x^{\gamma-1}(1-x)^{\alpha+\beta-\gamma}$. Whether or not this is a singular or regular problem depends on the values of α, β, γ. As long as $\gamma \neq 1$, r will vanish or explode at $x = 0$ and we will have a singular problem. If $\gamma = 1$, then p vanishes at $x = 0$. So this problem is always singular.

$$-[x^{\gamma}(1-x)^{1+\alpha+\beta-\gamma} \cdot y']' = \lambda x^{\gamma-1}(1-x)^{\alpha+\beta-\gamma} \cdot y$$

Super Bonus Problem: I'll grade this on a case by case basis...