Solutions to Homework # 7
Math 381, Rice University, Fall 2003

Problem 1: We work each by hand using integration by parts.

a) We integrate by parts twice to find that

$$\begin{aligned}
\mathcal{L}(e^{at}\cos(kt)) &= \int_0^\infty e^{-st}e^{at}\cos(kt)\ dt \\
&= \int_0^\infty e^{-(s-a)t}\cos(kt)\ dt \\
&= \frac{-e^{-(s-a)t}\cos(kt)}{s-a}\Big|_0^\infty - \int_0^\infty \frac{ke^{-(s-a)t}\sin(kt)}{s-a}\ dt \\
&= \frac{1}{s-a} - \frac{k}{s-a}\left(\int_0^\infty e^{-(s-a)t}\sin(kt)\ dt\right) \\
&= \frac{1}{s-a} - \frac{k}{s-a}\left(\frac{-e^{-(s-a)t}\sin(kt)}{s-a}\Big|_0^\infty + \int_0^\infty \frac{ke^{-(s-a)t}\cos(kt)}{s-a}\ dt\right) \\
&= \frac{1}{s-a} - \frac{k^2}{(s-a)^2}\mathcal{L}(e^{at}\cos(kt)).
\end{aligned}$$

We solve this equation to find $\mathcal{L}(e^{at}\cos(kt)) = \dfrac{s-a}{(s-a)^2+k^2}$.

b) Integrating by parts once implies that

$$\begin{aligned}
\mathcal{L}(t^n e^{-at}) &= \int_0^\infty e^{-st}t^n e^{-at}\ dt \\
&= \int_0^\infty e^{-(s+a)t}t^n\ dt \\
&= \frac{-t^n e^{-(s+a)t}}{s+a}\Big|_0^\infty + \int_0^\infty \frac{ne^{-(s+a)t}t^{n-1}}{s-a}\ dt \\
&= 0 + \frac{n}{s+a}\mathcal{L}(t^{n-1}e^{-at}).
\end{aligned}$$

Using this n times, decreasing the power of t each time, we see that

$$\mathcal{L}(t^n e^{-at}) = \frac{n!}{(s+a)^n}\int_0^\infty e^{-(s+a)t}\ dt = \frac{n!}{(s+a)^{n+1}}.$$

c) This one is similar to part (a), but the limits of integration turn out to be different.

$$\begin{aligned}
\mathcal{L}(f) &= \int_0^\infty f(t)e^{-st}\ dt \\
&= \int_0^\pi e^{-st}\sin(t)\ dt \\
&= e^{-s\pi} + 1 - s\int_0^\pi e^{-st}\cos(t)\ dt \\
&= e^{-s\pi} + 1 - s\left(0 + s\int_0^\pi e^{-st}\sin(t)\ dt\right) \\
&= e^{-s\pi} + 1 - s^2\mathcal{L}(f).
\end{aligned}$$

We then solve this equation to find that

$$\mathcal{L}(f) = \frac{e^{-s\pi}+1}{s^2+1}.$$

d) This one only requires one direct integration.

$$\begin{aligned}
\mathcal{L}(f) &= \int_0^\infty f(t)e^{-st}\ dt \\
&= \int_a^b e^{-st}\ dt \\
&= \frac{-1}{s}e^{-st}|_a^b = \frac{e^{-sa}-e^{-sb}}{s}.
\end{aligned}$$

Problem 2: For these we are free to use the results developed in class and in the book.

a) $\mathcal{L}(t^3) = \mathcal{L}(t^3\cdot 1) = (-1)^3\frac{d^3}{ds^3}\left(\frac{1}{s}\right) = \frac{6}{s^4}.$

b) $\mathcal{L}(t^2e^{-3t}) = (-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s+3}\right) = \frac{2}{(s+3)^3}.$

c) This one requires remembering that $\sinh(x) = (e^x - e^{-x})/2$.

$$\begin{aligned}
\mathcal{L}(\cos(at)\sinh(at)) &= \frac{1}{2}\mathcal{L}(\cos(at)(e^{at}-e^{-at})) \\
&= \frac{1}{2}\left(\mathcal{L}(e^{at}\cos(at)) - \mathcal{L}(e^{-at}\cos(at))\right) \\
&= \frac{1}{2}\left(\frac{s-a}{(s-a)^2+a^2} - \frac{s+a}{(s+a)^2+a^2}\right)
\end{aligned}$$

If you want, you can simplify this so that it matches the back of the book:

$$\mathcal{L}(\cos(at)\sinh(at)) = \frac{a(s^2-2a^2)}{s^4+4a^4}.$$

d) For this, recall that the rule about multiplying by an exponential implies that

$$\mathcal{L}(e^t \sin(2t)) = \frac{2}{(s-1)^2+4}.$$

So now using the rule about multiplying by t, we get

$$\begin{aligned}\mathcal{L}(te^t \sin(2t)) &= (-1)\frac{d}{ds}\left(\frac{2}{(s-1)^2+4}\right)\\ &= \frac{4(s-1)}{((s-1)+4)^2}.\end{aligned}$$

e) Again, this uses the rule about multiplying by t and some algebra.

$$\begin{aligned}\mathcal{L}(t^2 \sin(at)) &= (-1)^2\frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right)\\ &= \frac{6as^2-2a^3}{(s^2+a^2)^3}.\end{aligned}$$

f) And finally...

$$\mathcal{L}(e^{at}\cosh(bt)) = \frac{s-a}{(s-a)^2-b^2}.$$

Problem 3: For most of these we need to use a partial fraction decomposition.

a) Note that $\frac{1}{s^2-3s+2} = \frac{1}{s-2} - \frac{1}{s-1}$, so

$$\mathcal{L}^{-1}\left(\frac{1}{s^2-3s+2}\right) = e^{2t} - e^t.$$

b) We have, by completing the square,

$$\frac{1}{s^2-2s+5} = \frac{s}{(s-1)^2+4} = \frac{s-1}{(s-1)^2+4} + \frac{1}{(s-1)^2+4}.$$

So

$$\mathcal{L}^{-1}\left(\frac{1}{s^2-2s+5}\right) = e^t\cos(2t) + \frac{1}{2}e^t\sin(2t).$$

c) I got lots of different answers for this one before I came up with the answer the book claimed. This happens sometimes: there may be several equivalent ways to write the answer, but no easy way to see the equivalence. We see that

$$\frac{s+1}{s^4+1} = \frac{s}{s^4+1} + \frac{1}{s^4+1}.$$

Now if we choose $a = 1/\sqrt{2}$, then we can use several items on the transform table to get the answer the book lists. Specifically, you need to rewrite the above as

$$\frac{s+1}{s^4+1} = \frac{2a^2 s}{s^4+4a^4} + \frac{1}{\sqrt{2}}\frac{4a^3}{s^4+4a^4}.$$

So then

$$\mathcal{L}^{-1}\left(\frac{s+1}{s^4+1}\right) = \sin(at)\sinh(at) + \frac{1}{\sqrt{2}}[\sin(at)\cosh(at) - \cos(at)\sinh(at)].$$

Finally, note that $at = T = t/\sqrt{2}$, so we match...

d) Again by partial fractions, $\dfrac{2s+1}{s(s+1)(s+2)} = \dfrac{1/2}{s} + \dfrac{1}{s+1} - \dfrac{3/2}{s+2}$. Which means that

$$\mathcal{L}^{-1}\left(\frac{2s+1}{s(s+1)(s+2)}\right) = \frac{1}{2} + e^{-t} - \frac{3}{2}e^{-2t}.$$

e) Again by partial fractions, $\dfrac{1}{s^2(s^2+1)} = \dfrac{1}{s^2} - \dfrac{1}{s^2+1}$. This means that

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+1)}\right) = t - \sin(t).$$

f) This one only uses the "translation to exponential" property of the Laplace transform.

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{s+1}\right) = \begin{cases} 0, & 0 \le t \le 1 \\ e^{-(t-1)}, & t > 1 \end{cases}.$$

Problem 4:

a) We take the Laplace transform with respect to t.

$$-[y'(0) + sy(0)] + s^2\overline{y}(s) + 2[-y(0) + s\overline{y}(s)] + 2\overline{y} = 0.$$

Together with the initial conditions, we get

$$1 - s + s^2\overline{y} - 2 + 2 + 2s\overline{y} + 2\overline{y} = 0.$$

This equation has solution

$$\overline{y} = \frac{s+1}{(s+1)^2+1}.$$

Taking the inverse transform we find that

$$y(t) = e^{-t}\cos(t).$$

b) The Laplace transform of the equation is

$$-[y'(0)+sy(0)]+s^2\overline{y}+2[-y(0)+s\overline{y}]+2\overline{y}=2\frac{1}{s}.$$

Applying the initial condition, we get

$$-1+s^2\overline{y}+2s\overline{y}+2\overline{y}=\frac{2}{s}.$$

This equation has solution

$$\overline{y}=\frac{1}{s}-\frac{s+1}{(s+1)^2+1}.$$

Hence,

$$y(t)=1-e^{-t}\cos(t).$$

c) The Laplace transform of the equation is

$$-[y'(0)+sy(0)]+s^2\overline{y}(s)-2y(0)+2s\overline{y}(s)+2\overline{y}=e^{-s}.$$

With the initial conditions, we get

$$\overline{y}(s^2+2s+2)-(s+1)=e^{-s}.$$

We solve this algebraically to obtain

$$\overline{y}=\frac{e^{-s}}{(s+1)^2+1}+\frac{s+1}{(s+1)^2+1}.$$

Taking the inverse transform we find

$$\begin{aligned} y(t) &= \int_0^\infty \delta(t-u-1)e^{-u}\sin(u)\ du+e^{-t}\cos(t) \\ &= \begin{cases} 0, & 0\le t<1 \\ e^{-(t-1)}\sin(t-1)+e^{-t}\cos(t), & t\ge 1. \end{cases} \end{aligned}$$

d) Things proceed just like above, when we solve for $\overline{y}$ we get

$$\overline{y}=\frac{\overline{f}}{(s+1)^2+1}+\frac{y_0+y_0'}{(s+1)^2+1}+\frac{y_0(s+1)}{(s+1)^2+1}.$$

Taking the inverse transform, we get

$$y(t)=e^{-t}y_0\cos(t)+e^{-t}(y_0+y_0')\sin(t)+e^{-t}\int_0^t f(u)e^u\sin(t-u)\ du.$$

Problem 5: We take the Laplace transform with respect to t. After applying the initial conditions, we see that

$$\frac{\partial}{\partial x}\overline{z}(x,s) = (1-s)\overline{z}(x,s) + 1.$$

This is an inhomogeneous equation, but the inhomogeneity is a constant, so it is not difficult to use a guess and check approach or even another Laplace transform to find its solution. It is

$$\overline{z}(x,s) = \frac{e^{-s}}{s} - \frac{e^{-s}}{s-1} + \frac{1}{s-1}.$$

Now, we take the inverse transform. It is important to keep careful track of what happens in the "translation to exponential" step. We get

$$\begin{aligned} z(x,t) &= e^x \cdot \left\{ \begin{array}{ll} 0, & 0 \le t < x \\ 1, & t \ge x \end{array} \right\} - e^x \cdot \left\{ \begin{array}{ll} 0, & 0 \le t < x \\ e^{t-x}, & t \ge x \end{array} \right\} + e^t \\ &= \left\{ \begin{array}{ll} e^t, & 0 \le t < x \\ e^x, & t \ge x. \end{array} \right. \end{aligned}$$

Problem 6: Again, we use the Laplace transform on t. We get

$$\frac{\partial^2}{\partial x^2}\overline{\varphi}(x,s) + 1 + s - s^2\overline{\varphi} = \frac{1}{s}.$$

This is an inhomogeneous equation, where the inhomogeneous term is a constant. I used the guess and check method together with the solution to the homogeneous version to find

$$\overline{\varphi}(x,s) = A(s)e^{-sx} + B(s)e^{sx} + \frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^3}.$$

Since for really large values of x the second term cannot be a transform unless $B = 0$, we eliminate that term. Using the boundary condition $\overline{\varphi}(0,s) = 1/s$, we solve for A to find $A(s) = \frac{1}{s^3} - \frac{1}{s^2}$. This means that

$$\overline{\varphi}(x,s) = \frac{e^{-xs}}{s^3} - \frac{e^{-xs}}{s^2} + \frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^3}.$$

Finally, we take the inverse transform to find

$$\varphi(x,t) = 1 + t - \frac{1}{2}t^2 + \left\{ \begin{array}{ll} 0, & 0 \le t < x \\ \frac{1}{2}(t-x)^2, & t \ge x \end{array} \right\} - \left\{ \begin{array}{ll} 0, & 0 \le t < x \\ t-x, & t \ge x \end{array} \right\}$$

which simplifies to

$$\varphi(x,t) = \left\{ \begin{array}{ll} 1 + t + \frac{1}{2}, & 0 \le t < x \\ 1 - tx + x + \frac{1}{2}x^2, & t \ge x \end{array} \right\}$$

Problem 7: Using the method above but stopping before you take the inverse transform, it is not hard to find the equation the book quotes. It is even easier if you remember that the general solution of the equation $y_{xx} = a^2 y$ can be written as $y = A\sinh(ax) + B\cosh(ax)$. Applying the initial conditions finishes the first step.

The interesting part of the problem starts after the first equation. To see the next step, one has to use the rule about the transform of a derivative and the rule about the inverse transform of a product. We start by noting

$$\mathcal{L}\left(\frac{\partial}{\partial t}A(x,t)\right) = -A(x,0) + s\mathcal{L}(A(x,s)) = -A(x,0) + \frac{\sinh(sx/c)}{\sinh(sl/c)}.$$

The next important step is that $A(x,0) = 0$. This follows from the discussion we had in class about the way that a Laplace transform must decay (or from your text in chapter 2 during the same type of discussion) that $\lim_{s\to\infty} s\overline{A}(x,s) = A(x,0)$. But in our case, this limit is easy to evaluate for all $0 < x < l$.

So now, by the rule about inverse transforms of products, we see that

$$\varphi(x,t) = \int_0^t \frac{\partial A(x,t-u)}{\partial t} f(u)\ du.$$

The final statement now follows by using integration by parts.